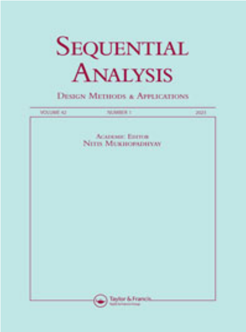

Sequential Analysis
Design Methods and Applications

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/lsqa20

On the exact constants in one-sided maximal inequalities for Bessel processes

Cloud Makasu

SEQUENTIAL ANALYSIS
2023, VOL. 42, NO. 1, 35–42
https://doi.org/10.1080/07474946.2022.2150778

Taylor & Francis
Taylor & Francis Group

On the exact constants in one-sided maximal inequalities for Bessel processes

Cloud Makasu

Department of Maths and Applied Maths, University of the Western Cape, Bellville, Cape Town

ABSTRACT
In this paper, we establish a one-sided maximal moment inequality with exact constants for Bessel processes. As a consequence, we obtain an exact constant in the Burkholder-Gundy inequality. The proof of our main result is based on a pure optimal stopping problem of the running maximum process for a Bessel process. The present results extend and complement a number of related results previously known in the literature.

ARTICLE HISTORY
Received 16 August 2022
Revised 21 October 2022
Accepted 26 October 2022

KEYWORDS
Bessel processes; Burkholder-Gundy inequalities; optimal stopping problem

1. INTRODUCTION

Let $X = (X_t)_{t\geq 0}$ be a Bessel process (see (Revuz and Yor, 1991)) with dimension $\alpha > 0$, starting at $x \geq 0$, which is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. It is well known that a Bessel process is a submartingale for $\alpha \geq 1$, and is a supermartingale when $\alpha \leq 0$. For the case $0 < \alpha < 1$, the Bessel process is not a semimartingale. Throughout the paper, we assume that $X = (X_t)_{t\geq 0}$ is given by

$$dX_t = \frac{\alpha - 1}{2X_t}\ dt + dB_t; \quad X(0) = x, \tag{1}$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion under $\mathbf{P}$.

Define $X_t^* = \sup_{0\leq r\leq t} X_r$. For Bessel processes, the Burkholder-Gundy type of inequalities (Burkholder, 1977) are established in (DeBlassie, 1987) without exact constants, see also Theorem 4.1 in (Graversen and Peskir, 1998). A special case of these inequalities (DeBlassie, 1987) asserts that there exist constants c_α and C_α depending only on α such that

$$c_\alpha \mathbf{E}_x \sqrt{\tau + x^2} \leq \mathbf{E}_x\left[X_\tau^*\right] \leq C_\alpha \mathbf{E}_x \sqrt{\tau + x^2} \tag{2}$$

for any stopping time τ of X starting at $x \geq 0$. On the other hand, it is proved in (Dubins et al., 1994) that there exists a constant A_α such that

$$\mathbf{E}\left[X_\tau^*\right] \leq A_\alpha \sqrt{\mathbf{E}[\tau]} \tag{3}$$

for arbitrary stopping times τ of a Bessel process starting at zero.

CONTACT Cloud Makasu cmakasu@uwc.ac.za Dept of Maths and Applied Maths, University of the Western Cape, Private Bag X17, Bellville 7535, Cape Town

Our motivation is the proof of the maximal moment inequality (3) given in (Dubins et al., 1994), and the question arising on the exact constants for c_α and C_α in the Burkholder-Gundy inequality (2). The main purpose of this paper is to establish the exact form of c_α in (2) for a Bessel process $X = (X_t)_{t\geq 0}$ with dimension strictly $1 \leq \alpha < 2$. This is given in Corollary 2.1 as a consequence of our main result stated in Theorem 1.1. Our proof rests on the optimal stopping theory of the maximum process for Bessel processes (Dubins et al., 1994), see also (Dubins and Schwarz, 1988) for the Brownian motion case and (Peskir, 1998) for general diffusion processes. A variant of the optimal stopping problems treated in (Dubins and Schwarz, 1988) and (Dubins et al., 1994) is considered here. In the present case, the integral cost in (6) only depends on the running maximum process associated with the Bessel process. The motivation will be apparent in the proof of our main result.

More precisely, we shall prove the following result in this paper. A consequence of this result will be proved in Corollary 2.1.

Theorem 1.1. *Let $X = (X_t)_{t\geq 0}$ be a Bessel process given by (1) with dimension $1 \leq \alpha < 2$ fixed, and starting at $x \geq 0$. Then,*

$$\sqrt{\alpha}\ \mathbf{E}_x[\tau^{1/2}] + \frac{\alpha^2}{1+\alpha^2}\left(2 - (\alpha+1)(\alpha-1)^{\frac{\alpha}{2-\alpha}}\right)x \leq \mathbf{E}_x\left[\sup_{0\leq t\leq \tau} X_t\right] \tag{4}$$

for any stopping time τ of X.

For the proof of the above result, we shall provide the necessary details only for instances which demand different arguments from those given in (Dubins et al., 1994). The use of converse Hölder's inequality and the reverse Young inequality will play an important role in our proof. This is not the case in (Dubins et al., 1994) where the proof of (3) is based on maximizing an expected payoff with a constant cost of observation. However, we note that the construction of the value function (8) and optimal stopping time (9) in the present case (6) follows from a modification of the arguments in (Dubins et al., 1994). We further employ a comparison principle (see (Lakshmikantham, 1962), Theorem 1) in our proof to establish the maximal property of the optimal stopping boundary in question.

2. PROOF OF THE THEOREM

Proof. Let $X = (X_t)_{t\geq 0}$ be a Bessel process given by (1). Fix $\alpha \geq 1$, and assume that $\alpha \neq 2$ only for simplicity. Define

$$Y_t = \left(\sup_{0\leq r\leq t} X_r\right)\vee y \tag{5}$$

for all $0 \leq x \leq y$.

Now consider the following optimal stopping problem:

$$u(x,y) := \inf_{\tau} \mathbf{E}_{x,y}\left[Y_\tau - c\int_0^\tau \frac{1}{Y_s}\ ds\right], \tag{6}$$

where the infimum is taken over all stopping times τ for X such that the integral in (6) has finite expectation, $\mathbf{E}_{x,y}$ denotes the expectation with respect to the probability law $\mathbf{P}_{x,y} := \mathbf{P}$ of the process (X, Y) starting at (x, y) with $0 \leq x \leq y$, and $c > 0$ is some constant. In what follows, we shall show that the positive constant c is such that

$$\frac{\alpha}{1+(\alpha-1)^{\frac{2}{2-\alpha}}} < c < \alpha \tag{7}$$

for $1 < \alpha < 2$. We shall treat the particular case $\alpha = 1$ separately in the course of the proof.

Using similar arguments as in (Dubins et al., 1994) by modifying Eq. (3.10), using the continuity condition (3.16), smooth-fit principle (3.17) and normal reflection boundary (3.18), it follows that

$$u(x,y) = \begin{cases} y + \frac{c}{y}\left(\frac{1}{2-\alpha}g_*^2(y) - \frac{2}{\alpha(2-\alpha)}x^{2-\alpha}g_*^\alpha(y) + \frac{1}{\alpha}x^2\right); & \text{if } g_*(y) \le x \le y \\ y; \quad \text{if } 0 \le x \le g_*(y). \end{cases} \tag{8}$$

On the other hand, we have the optimal stopping time τ^* of the form

$$\tau^* = \inf\{t > 0 | X_t \le g_*(Y_t)\}, \tag{9}$$

where by the maximality principle (Peskir, 1998) the stopping boundary $y \mapsto g_*(y)$ is the maximal solution of the first-order nonlinear differential equation

$$g'(y) = \frac{\frac{y}{cg(y)} - \frac{g(y)}{y}\left(\frac{1}{\alpha}\left(\frac{y}{g(y)}\right)^2 + \frac{1}{2-\alpha} - \frac{2}{\alpha(2-\alpha)}\left(\frac{g(y)}{y}\right)^{\alpha-2}\right)}{\frac{2}{2-\alpha}\left(\left(\frac{g(y)}{y}\right)^{\alpha-2} - 1\right)} \tag{10}$$

under the condition $0 < g(y) < y$.

For the first part of the proof, we establish an upper estimate for any positive solution $y \mapsto g(y)$ of Eq. (10). We shall need this estimate to show that $y \mapsto g_*(y)$ is the maximal solution. The proof rests on a comparison principle argument (see (Lakshmikantham, 1962; Szarski, 1965; Walter, 1964), etc). Let Δ denote an open set $\Delta = \{(y,g) | 0 < g(y) < y\}$ and let $\mathcal{X}(y, g(y))$ denote the right-hand side of (10). It can be easily shown that $\mathcal{X}(y, g(y))$ is continuous, and

$$|\mathcal{X}(y,g(y))| \le \frac{\left|\frac{1}{c} - \frac{1}{\alpha}\right|\frac{y}{g(y)} + \frac{1}{|2-\alpha|}\left|\frac{2}{\alpha}\left(\frac{g(y)}{y}\right)^{\alpha-1} - \frac{g(y)}{y}\right|}{\frac{2}{|2-\alpha|}\left|\left(\frac{g(y)}{y}\right)^{\alpha-2} - 1\right|} \tag{11}$$

on the open set Δ. This immediately implies that there exists a maximal solution for Eq. (10). Consider the following first-order nonlinear differential equation

$$h'(y) = \frac{\left|\frac{1}{c} - \frac{1}{\alpha}\right|\frac{y}{h(y)} + \frac{1}{|2-\alpha|}\left|\frac{2}{\alpha}\left(\frac{h(y)}{y}\right)^{\alpha-1} - \frac{h(y)}{y}\right|}{\frac{2}{|2-\alpha|}\left|\left(\frac{h(y)}{y}\right)^{\alpha-2} - 1\right|} \tag{12}$$

on the common interval of existence with (10), and assume that $h(0) = |g(0)|$ here and below.

Its clear that

$$h(y) = \beta y, \quad (y > 0) \tag{13}$$

is a positive solution of (12), where $0 < \beta < 1$ and is the maximal root that satisfies

$$\frac{2}{|2-\alpha|}|\beta^\alpha - \beta^2| - \frac{1}{|2-\alpha|}\left|\frac{2}{\alpha}\beta^\alpha - \beta^2\right| - \left|\frac{1}{c} - \frac{1}{\alpha}\right| = 0. \tag{14}$$

Then, by a comparison principle argument (see (Lakshmikantham, 1962) for instance, Theorem 1), we have the estimate

$$g(y) \le \beta y \tag{15}$$

for any positive solution $y \mapsto g(y)$ of (10). On the other hand, we have

$$g_*(y) = \theta y, \quad (y > 0) \tag{16}$$

as a positive solution of (10) with $0 < \theta < 1$ and being the maximal root of the equation

$$\frac{2}{2-\alpha}\left(1 - \frac{1}{\alpha}\right)\theta^\alpha - \frac{1}{2-\alpha}\theta^2 - \frac{1}{c} + \frac{1}{\alpha} = 0. \tag{17}$$

By definition, $y \mapsto g_*(y)$ is a maximal solution of Eq. (10) if $g(y) \le g_*(y)$ for any positive solution $g(y)$ of (10). We shall first show that (16) is a maximal solution in the particular case $\alpha = 1$. This will follow from the estimate (15) and by the definition once it is shown that $\beta = \theta$. Assume that $\alpha = 1$ in (17). Then, immediately we have the maximal root

$$\theta = \sqrt{1 - \frac{1}{c}} \tag{18}$$

provided that $c > 1$. Similarly, if $\alpha = 1$ in (14), then it follows that

$$2|\beta - \beta^2| - |2\beta - \beta^2| - \left|\frac{1}{c} - 1\right| = 0. \tag{19}$$

Suppose that the terms under the modulus signs are negative. Therefore,

$$\beta^2 + \frac{1}{c} - 1 = 0 \tag{20}$$

for $c > 1$. Now choose β in (20) such that $\beta = \sqrt{1 - \frac{1}{c}}$ for $c > 1$. Consequently, $\beta = \theta$ and it follows from (15) that $g(y) \le \theta y$ for any positive solution $y \mapsto g(y)$ of (10). Hence, we have just proved that $g_*(y) = \theta y$ is the maximal solution of (10) with $\theta = \sqrt{1 - \frac{1}{c}}$ when $c > 1$ and in the particular case $\alpha = 1$. In what follows, we shall give the proof in the remaining case when $1 < \alpha < 2$. We first show the inequality (7). Let $f(\theta)$ be a real-valued function which is continuous on a closed interval $[0, 1]$ and differentiable on (0, 1) defined by

$$f(\theta) := \frac{2}{2-\alpha}\left(1 - \frac{1}{\alpha}\right)\theta^\alpha - \frac{1}{2-\alpha}\theta^2 - \frac{1}{c} + \frac{1}{\alpha}. \tag{21}$$

Clearly, $f(0) = -\frac{1}{c} + \frac{1}{\alpha}$ and $f(1) = -\frac{1}{c}$. Now assume that $f(0) < 0$. Then, using an elementary argument, there exists a maximal root θ satisfying (17) such that

$$(\alpha - 1)^{\frac{1}{2-\alpha}} < \theta \tag{22}$$

if and only if c is a positive constant satisfying the restriction

$$c > \frac{\alpha}{1 + (\alpha - 1)^{\frac{2}{2-\alpha}}} \tag{23}$$

for $1 < \alpha < 2$. Assuming that $f(0) < 0$ and using (23), for $1 < \alpha < 2$, then we have

$$\frac{\alpha}{1 + (\alpha - 1)^{\frac{2}{2-\alpha}}} < c < \alpha \tag{24}$$

which proves (7).

Using the estimate in (15), we shall now show that $g_*(y) = \theta y$ in (16) is the maximal solution of (10) in the case $1 < \alpha < 2$. Let θ be the maximal root of (17) for $1 < \alpha < 2$. For $0 < \beta < 1$ and using the fact that $c < \alpha$ for $1 < \alpha < 2$, it follows from (14) that

$$\frac{2}{2-\alpha}\left(1 - \frac{1}{\alpha}\right)\beta^{\alpha} - \frac{1}{2-\alpha}\beta^2 - \frac{1}{c} + \frac{1}{\alpha} = 0. \tag{25}$$

Hence, we deduce immediately from (17) and (25) that the maximal roots are such that $\theta = \beta$. This and the estimate (15) prove the maximal property of the solution $y \mapsto g_*(y)$ in the remaining case $1 < \alpha < 2$. We have completed the proof of the first part of the theorem.

Let ν and ρ denote the Hölder exponents such that $\frac{1}{\nu} + \frac{1}{\rho} = 1$ with $\nu < 0$. Now using the reverse Hölder inequality, it follows from (6) that

$$\begin{aligned} \mathbf{E}_{x,y}[Y_\tau] &\geq c\mathbf{E}_{x,y}\left[\int_0^\tau \frac{1}{Y_s}\ ds\right] + u(x,y) \\ &\geq c\mathbf{E}_{x,y}\left[Y_\tau^{-1}\tau\right] + u(x,y) \\ &\geq \left(\mathbf{E}_{x,y}[Y_\tau^{-\nu}]\right)^{1/\nu}\left(c^\rho \mathbf{E}_{x,y}[\tau^\rho]\right)^{1/\rho} + u(x,y). \end{aligned} \tag{26}$$

Then, applying the reverse Young inequality in (26), we have

$$\mathbf{E}_{x,y}[Y_\tau] \geq \frac{1}{\nu}\mathbf{E}_{x,y}[Y_\tau^{-\nu}] + \frac{1}{\rho}c^\rho \mathbf{E}_{x,y}[\tau^\rho] + u(x,y). \tag{27}$$

Now choose $\nu = -1$ so that $\rho = \frac{1}{2}$. Hence,

$$\mathbf{E}_{x,y}[Y_\tau] \geq \sqrt{c}\mathbf{E}_{x,y}[\tau^{1/2}] + \frac{1}{2}u(x,y) \tag{28}$$

which follows immediately from (27).

On the other hand, we have

$$\begin{aligned} \mathbf{E}_{x,y}[Y_\tau] &\geq \left(c^{\nu/\rho}\mathbf{E}_{x,y}[Y_\tau^{-\nu}]\right)^{1/\nu}\left(c^{\rho/\nu}\mathbf{E}_{x,y}[\tau^\rho]\right)^{1/\rho} + u(x,y) \\ &\geq \frac{1}{\nu}c^{\nu/\rho}\mathbf{E}_{x,y}[Y_\tau^{-\nu}] + \frac{1}{\rho}c^{\rho/\nu}\mathbf{E}_{x,y}[\tau^\rho] + u(x,y) \end{aligned} \tag{29}$$

which follows from (26) and by using the reverse Young inequality.

Now arguing similarly by choosing $\nu = -1$ and $\rho = \frac{1}{2}$ in (29) and followed by a simple re-arrangement, we obtain

$$\mathbf{E}_{x,y}[Y_\tau] \geq \frac{2c^2}{\sqrt{c}(1+c^2)}\mathbf{E}_{x,y}[\tau^{1/2}] + \frac{c^2}{1+c^2}u(x,y). \tag{30}$$

It is easy to show from (26) that

$$\mathbf{E}_{x,y}[Y_\tau] \geq \frac{2c}{1+c}\mathbf{E}_{x,y}[\tau^{1/2}] + \frac{1}{1+c}u(x,y) \tag{31}$$

and

$$\mathbf{E}_{x,y}[Y_\tau] \geq \frac{2c}{1+c}\mathbf{E}_{x,y}[\tau^{1/2}] + \frac{c}{1+c}u(x,y) \tag{32}$$

by choice of $\nu = -1$ and $\rho = \frac{1}{2}$.

Using (8) and (16), we have

$$u(x,x) = x\left(1 + c\left(\frac{\theta^2}{2-\alpha} - \frac{2\theta^\alpha}{\alpha(2-\alpha)} + \frac{1}{\alpha}\right)\right). \tag{33}$$

The desired result (4) now follows by letting $x = y$ in (28, 30, 31) and (32), using (33) and passing to the limit as $c \uparrow \alpha$ and $\theta \downarrow (\alpha-1)^{\frac{1}{2-\alpha}}$ for $1 \leq \alpha < 2$. The proof of the theorem is now complete. □

Remark 1. It should be noted that the upper estimate in (24) is not possible when the integral cost $\int_0^\tau \frac{1}{Y_s}\, ds$ in (6) is replaced by $\int_0^\tau \frac{1}{X_s}\, ds$. The upper estimate is crucial in the proof of our Theorem 1.1. We further remark that the case with the integral cost $\int_0^\tau \frac{1}{X_s}\, ds$ requires a modification of the value function $u(x, y)$ in (8). The details are left to the interested reader.

The following result and its proof is a consequence of Theorem 1.1. We now establish the exact form of the constant c_α in (2) in this result.

Corollary 2.1. Assume that $X = (X_t)_{t\geq 0}$ is a Bessel process given by (1) starting at $x \geq 0,$ and with dimension $1 \leq \alpha < 2$ fixed. Then, we have

$$\frac{\alpha^2}{1+\alpha^2}\left(2 - (\alpha+1)(\alpha-1)^{\frac{\alpha}{2-\alpha}}\right)\mathbf{E}_x\sqrt{\tau + x^2} \leq \mathbf{E}_x\left[\sup_{0\leq t\leq \tau} X_t\right] \tag{34}$$

for any stopping time τ of X.

Proof. This follows immediately from Theorem 1.1 and using an elementary inequality. Let a, b be non-negative real numbers and $0 < r \leq 1,$ then it follows that

$$(a+b)^r \leq a^r + b^r. \tag{35}$$

Now for any $1 \leq \alpha < 2,$ its easy to see from (4) that

$$
\begin{aligned}
\mathbf{E}_x\Big[\sup_{0\le t\le\tau} X_t\Big] &\ge \sqrt{\alpha}\mathbf{E}_x[\tau^{1/2}] + \frac{\alpha^2}{1+\alpha^2}\Big(2-(\alpha+1)(\alpha-1)^{\frac{\alpha}{2-\alpha}}\Big)x \\
&\ge \frac{\alpha^2}{1+\alpha^2}\Big(2-(\alpha+1)(\alpha-1)^{\frac{\alpha}{2-\alpha}}\Big)\mathbf{E}_x\big[\tau^{1/2}+x\big] \\
&= \frac{\alpha^2}{1+\alpha^2}\Big(2-(\alpha+1)(\alpha-1)^{\frac{\alpha}{2-\alpha}}\Big)\mathbf{E}_x\Big[\tau^{1/2}+(x^2)^{1/2}\Big] \\
&\ge \frac{\alpha^2}{1+\alpha^2}\Big(2-(\alpha+1)(\alpha-1)^{\frac{\alpha}{2-\alpha}}\Big)\mathbf{E}_x\sqrt{\tau+x^2},
\end{aligned}
\tag{36}
$$

where the last inequality follows using (35) with $r=\frac{1}{2}$. This completes the proof. □

Remark 2. At point $x=0$, we deduce from (4) and (34) that

$$
\sqrt{\alpha}\ \mathbf{E}[\tau^{1/2}] \le \mathbf{E}\Big[\sup_{0\le t\le\tau} X_t\Big] \tag{37}
$$

which is a one-sided Burkholder-Gundy inequality with an exact constant.

Remark 3. For the particular case when $\alpha=2$ in Theorem 1.1 and Corollary 2.1, we simply replace $(\alpha-1)^{\frac{\alpha}{2-\alpha}}$ by $\frac{1}{e^2}$. Hence, we have

$$
\sqrt{2}\ \mathbf{E}_x[\tau^{1/2}] + \frac{4}{5}\Big(2-\frac{3}{e^2}\Big)x \le \mathbf{E}_x\Big[\sup_{0\le t\le\tau} X_t\Big] \tag{38}
$$

from the inequality (4) and

$$
\frac{4}{5}\Big(2-\frac{3}{e^2}\Big)\mathbf{E}_x\sqrt{\tau+x^2} \le \mathbf{E}_x\Big[\sup_{0\le t\le\tau} X_t\Big] \tag{39}
$$

which follows from (34) for a Bessel process $X=(X_t)_{t\ge0}$ with dimension $\alpha=2$ and starting at $x\ge0$. It follows immediately from (38) and (39) that

$$
\sqrt{2}\ \mathbf{E}[\tau^{1/2}] \le \mathbf{E}\Big[\sup_{0\le t\le\tau} X_t\Big]
$$

for a Bessel process starting at zero.

Remark 4. It is not immediate whether the present method of proof can be modified to cover the remaining case C_α in the upper bound in (2).

ACKNOWLEDGEMENT

The author is indebted to the Editor, Associate Editor and an anonymous referee for their useful comments and suggestions which have considerably improved the presentation of this paper.

DISCLOSURE STATEMENT

No potential conflict of interest was reported by the author.

REFERENCES

Burkholder, D. L. 1977. "Exit Times of Brownian Motion, Harmonic Majorization, and Hardy Spaces." *Advances in Mathematics* 26 (2):182–205.

DeBlassie, R. D. 1987. "Stopping Times of Bessel Processes." *The Annals of Probability* 15 (3): 1044–51.

Dubins, L. E., and G. Schwarz. 1988. "A Sharp Inequality for Submartingales and Stopping Times." *Astérisque* 157–158:129–45.

Dubins, L. E., L. A. Shepp, and A. N. Shiryaev. 1994. "Optimal Stopping Rules and Maximal Inequalities for Bessel Processes." *Theory of Probability & Its Applications* 38 (2):226–61.

Graversen, S. E., and G. Peskir. 1998. "Maximal Inequalities for Bessel Processes." *Journal of Inequalities and Applications* (2):621735.

Lakshmikantham, V. 1962. "Upper and Lower Bounds of the Norm of Solutions of Differential Equations." *Proceedings of the American Mathematical Society* 13 (4):615–6.

Peskir, G. 1998. "Optimal Stopping of the Maximum Process: The Maximality Principle." *The Annals of Probability* 26 (4):1614–40.

Revuz, D., and M. Yor. 1991. *Continuous Martingales and Brownian Motion*. New York: Springer-Verlag.

Szarski, J. 1965. *Differential Inequalities*. Warszawa: Polish Scientific Publishers.

Walter, W. 1964. *Differential and Integral Inequalities*. Berlin: Springer-Verlag.