AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

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N.B. Mumba & E. Mwambene

To cite this article: N.B. Mumba & E. Mwambene (2018) Automorphism groups of graph covers and uniform subset graphs, AKCE International Journal of Graphs and Combinatorics, 15:1, 27-30, DOI: 10.1016/j.akcej.2018.01.016

To link to this article: https://doi.org/10.1016/j.akcej.2018.01.016

Published online: 10 Jun 2020.

Available online at www.sciencedirect.com
ScienceDirect
AKCE International Journal of Graphs and Combinatorics 15 (2018) 27–30
AKCE International Journal of Graphs and Combinatorics
www.elsevier.com/locate/akcej

Automorphism groups of graph covers and uniform subset graphs

N.B. Mumba*, E. Mwambene

Department of Mathematics and Applied Mathematics, University of the Western Cape, Private Bag X17, Bellville 7535, South Africa

Received 27 January 2017; received in revised form 19 January 2018; accepted 22 January 2018
Available online 13 February 2018

Abstract

Hofmeister considered the automorphism groups of antipodal graphs through the exploration of graph covers. In this note we extend the exploration of automorphism groups of distance preserving graph covers. We apply the technique of graph covers to determine the automorphism groups of uniform subset graphs $\Gamma(2k, k, k-1)$ and $\Gamma(2k, k, 1)$. The determination of automorphism groups answers a conjecture posed by Mark Ramras and Elizabeth Donovan. They conjectured that $\mathrm{Aut}(\Gamma(2k, k, k-1)) \cong S_{2k} \times < T >$, where T is the complementation map $X \mapsto T(X) = X^c = \{1, 2, \ldots, 2k\} \setminus X$, and X is a k-subset of $\Omega = \{1, 2, \ldots, 2k\}$.

Keywords: Graph covers; Antipodal graphs; Uniform subset graphs

1. Introduction

In this paper, all graphs considered are assumed to be finite and simple. For a graph Γ, we let $V(\Gamma)$, $E(\Gamma)$, $\mathrm{Aut}(\Gamma)$ and $e = \{x, y\}$ denote the vertex set, the edge set, the automorphism group and an edge with endpoints x and y, respectively. A graph Γ is **vertex-transitive** if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. For a graph Γ, $x \in V(\Gamma)$, we set $\Gamma_i(x) := \{y \in V(\Gamma) | d(x, y) = i\}$ and $\epsilon(x) = \max\{d(x, y) | y \in V(\Gamma)\}$, where d is the usual shortest distance path in Γ. Let $\mathcal{P}$ be a partition of $V(\Gamma)$. By the quotient graph $\Gamma/\mathcal{P}$ is meant the graph with

$$V(\Gamma/\mathcal{P}) := \mathcal{P};$$

$$\{X, Y\} \in E(\Gamma/\mathcal{P}) \iff X \neq Y \text{ and } \{x, y\} \in E(\Gamma) \text{ for some } x \in X, y \in Y.$$

A graph Γ is said to be **antipodal** if the collection of sets $\{x\} \cup \Gamma_{\epsilon(x)}(x)$ is a partition of $V(\Gamma)$.

For a graph Γ and $A \subset V(\Gamma)$, the minimum distance of A is defined by $d(A) := \min_{x,y,x \neq y \in A} d(x, y)$.

Chen and Lih [1] introduced uniform subset graphs. This is a generalisation of such graphs as Kneser graphs and Johnson graphs. They are defined by the following. Let n, k be positive integers such that $n \geq 2k$, and i a non-negative

Peer review under responsibility of Kalasalingam University.
* Corresponding author.
E-mail addresses: nephtale@aims.ac.za (N.B. Mumba), emwambene@uwc.ac.za (E. Mwambene).

https://doi.org/10.1016/j.akcej.2018.01.016

integer with $i < k$. Let $\Omega = \{1, 2, \dots, n\}$ and $\Omega^{\{k\}}$ the set of all k-subsets of Ω. A **uniform subset graph** $\Gamma(n, k, i)$ is defined by

$$V(\Gamma(n,k,i)) = \Omega^{\{k\}};$$
$$\{X, Y\} \in E(\Gamma(n,k,i)) \iff |X \cap Y| = i, \ X, Y \in \Omega^{\{k\}}.$$

Let Ω be a non-empty set. S_Ω denotes the set of permutations of the set Ω. If $\Omega = \{1, 2, \dots, n\}$, S_Ω is written S_n. Since S_n is k-transitive for all $k \le n$ and preserves the size of intersection sets, it is easy to see that uniform subset graphs are vertex transitive. However, it is surprisingly difficult to determine their full automorphism groups. Amongst the many classes of uniform subset graphs it has only been determined that $\mathrm{Aut}(\Gamma(2k+1, k, 0))$, the automorphism group of the so called Odd graphs, is S_n, and more recently, Ramras and Donovan [2] proved that $\mathrm{Aut}(\Gamma(n, k, k-1))$, $n \ne 2k$ coincides with S_n. Further, they conjectured that $\mathrm{Aut}(\Gamma(2k, k, k-1)) \cong S_n \times \langle T \rangle$, where T is the complementation map $X \mapsto T(X) = X^c = \{1, 2, \dots, n\} \setminus X$, and $X \in \Omega^{\{k\}}$.

In this note we determine the automorphism groups of the uniform subset graphs $\Gamma(2k, k, 1)$ and $\Gamma(2k, k, k-1)$.

2. Automorphisms of graph covers

In order to determine the automorphism groups of the graphs in question, we employ Hofmeister's [3] strategy. He determines the automorphism group of a graph cover by first looking at the quotient (folded) graph. The key observation in analysing the automorphism group of the cover is in understanding the interplay between automorphisms of the cover and their corresponding quotient.

Definition 1. Let Γ and Δ be graphs and $r \ge 2$ be an integer. Δ is called an r-cover of Γ if there is an epimorphism $\rho : \Delta \to \Gamma$, called the covering projection, such that

(i) $|\rho^{-1}(x)| = r$, for every $x \in V(\Gamma)$;
(ii) ρ bijectively sends $\Delta_1(x)$ to $\Gamma_1(\rho(x))$ for each $x \in V(\Gamma)$.

The graph Γ is called the **fold** of Δ.

Gross and Tucker [4] have shown that graph covers arise from permutation voltage graphs so that the consideration of the former amounts to focusing on the later. Permutation voltage graphs are defined in the following.

For a graph Γ, let $A(\Gamma)$ be the arc set of the corresponding symmetric digraph. A permutation voltage assignment in a symmetric group S_r for Γ is a mapping $f : A(\Gamma) \to S_r$ such that $f((x, y)) = (f((y, x)))^{-1}$, for any arc (x, y) in $A(\Gamma)$.

Given a graph Γ and a permutation voltage assignment f, the derived graph Γ_f is the graph with

$$V(\Gamma_f) = V(\Gamma) \times \{1, 2, \dots, r\};$$
$$\{(x, i), (y, j)\} \in E(\Gamma_f) \iff \{x, y\} \in E(\Gamma), (f(x, y))i = j.$$

Gross and Tucker's reductions, as alluded to, is the content of the following two results.

Theorem 1 ([4]). *Let Γ be a graph and $f : A(\Gamma) \to S_r$ a permutation voltage assignment. Then the natural projection $\rho_f : \Gamma_f \to \Gamma$ (sending vertex (u, i) of Γ_f to vertex u of Γ) is an r-fold covering projection.*

Theorem 2 ([4]). *Let $\rho : \Delta \to \Gamma$ be a r-fold covering projection. Then there is an assignment f of voltages in the symmetric group S_r for Γ such that the covering projections ρ and ρ_f are isomorphic with respect to the trivial automorphism.*

It is in the context of permutation voltage graphs that Hofmeister [3] considered the interplay between the automorphisms of graph covers and their corresponding quotient graphs. The essence of this interplay is in the following. Let $H \le \mathrm{Aut}(\Gamma)$ be a group of automorphisms of the graph Γ. A H-automorphism of a covering projection $\rho : \Delta \to \Gamma$ is a pair (φ, ψ), consisting of an automorphism $\varphi \in H$ and an automorphism $\psi : \Delta \to \Delta$, such that $\varphi\rho = \rho\psi$.

As a generalisation, we consider r-coverings that are defined by semi-regular automorphisms in vertex-transitive graphs. A semi-regular element of an automorphism group of a graph is a non-identity element having all cycles of equal length in its cycle decomposition. In case of vertex transitive graphs admitting a semi-regular automorphisms, the following generalises antipodal coverings.

Lemma 1. *Let Γ be a vertex transitive graph. Let σ be a semi-regular automorphism of Γ such that the orbits of σ partition $V(\Gamma)$ in such a way that for each orbit of the automorphism, $X, d(X) > 2$. Then the natural projection $f : \Gamma \to \Gamma/\sigma$ defined by $x \mapsto X, x \in X$, is a covering, where Γ/σ is the partition induced by σ.*

Proof. That $|f^{-1}(x)| = r$, for some fixed positive integer r, is an immediate consequence of semi-regularity of σ.

For any $y, z \in N(x), y \neq z$, the orbit of the automorphism containing y is distinct from the orbit of the automorphism containing z, since $d(X) \geq 2$. Therefore $|N(X)| \geq |N(x)|$ and by vertex transitivity $|N(X)| = |N(x)|$. □

The critical issue Hofmeister [3] observed is the result we generalise to graph covers in Lemma 1. In this context, the result below follows from the fact that graph automorphisms are distance preserving.

Theorem 3. *Let Γ be a vertex transitive graph. Let σ be a semi-regular automorphism of Γ such that the orbits of σ partition $V(\Gamma)$ in such a way that for each orbit of the automorphism, $X, d(X) > 2$. Let $\rho : \Gamma \to \Gamma/\sigma$ be a covering projection and ψ an automorphism of Γ. Then there is an automorphism φ of Γ/σ such that $\varphi\rho = \rho\psi$, where Γ/σ is the quotient induced by σ.*

Following Hofmeister [3], we let the group $S_r^{V(\Gamma)}$ act on the set of permutation voltage assignments in S_r for any Γ by $\Pi(f(x, y)) = \pi_y^{-1} f(x, y)\pi_x$ when $\Pi = (\pi_u)_{u\in V(\Gamma)}$. The stabiliser of f under this action will be denoted by $\mathrm{Fix}(\Gamma, f)$.

In this generalised context, we recoup Hofmeister's characterisation of the automorphism groups of graph covers with respect to those of the corresponding quotient graphs and furthermore the argument of the proof of Theorem 5 in [3] holds.

Theorem 4. *Let Γ be a vertex transitive graph. Let σ be a semi-regular automorphism of Γ such that the orbits of σ partition $V(\Gamma)$ in such a way that for each orbit of the automorphism, $X, d(X) > 2$. Then there is a short exact sequence*

$$1 \to \mathrm{Fix}(\Gamma, f) \to \mathrm{Aut}(\Gamma_f) \to \mathrm{Aut}(\Gamma)_f \to 1.$$

3. Automorphisms of $\Gamma(2k, k, k-1)$ and $\Gamma(2k, k, 1)$

We now show that the graphs $\Gamma(2k, k, k-1)$ and $\Gamma(2k, k, 1)$ are covers of a graph $\Gamma(2k, k, i)/\sigma$, of which we can determine its automorphism group. Using Theorem 4, we therefore determine automorphism groups of $\Gamma(2k, k, k-1)$ and $\Gamma(2k, k, 1)$.

Now, consider $\sigma : \Gamma(2k, k, i) \to \Gamma(2k, k, i), i = 1, k-1$ defined by

$$\sigma(X) = \Omega \setminus X. \tag{1}$$

It is easy to see that σ is an automorphism of $\Gamma(2k, k, i)$. Moreover, $d(X, \Omega \setminus X) = \epsilon(X)$ if $i = k-1$ and $d(X, \Omega \setminus X) = 3$ if $i = 1$. Hence we have the following.

Corollary 1. *$\Gamma(2k, k, i), i = 1, k-1$ are covers of $\Gamma(2k, k, i)/\sigma$.*

To determine automorphism groups of $\Gamma(2k, k, k-1)$ and $\Gamma(2k, k, 1)$, it is enough to work out the automorphism groups of $\Gamma(2k, k, i)/\sigma, i = 1, k-1$.

Now, as for $\Gamma(2k, k, i)/\sigma, i = 1, k-1$, we have the following.

Lemma 2. *Let σ be the map defined in Eq. (1). Then*

(i) *$S_{2k} \cong$ (a subgroup of)* $\mathrm{Aut}(\Gamma(2k, k, i)/\sigma), i = 1, k-1$.
(ii) *$\Gamma(2k, k, i)/\sigma, i = 1, k-1$ is vertex transitive.*

Proof. (i) Let $X = \{A, B\} \in V(\Gamma(2k, k, i)/\sigma)$. The symmetric group S_{2k} acts on $V(\Gamma(2k, k, i)/\sigma)$ by $\theta(X) = \{\theta(A), \theta(B)\}$, where $\theta \in S_{2k}$. Moreover θ preserves the size of intersections. Hence $S_{2k} \cong$ (a subgroup of) $\mathrm{Aut}(\Gamma(2k, k, i)/\sigma)$, $i = 1, k-1$.

(ii) Let $\{\{a_1, a_2, \ldots, a_k\}, \{b_1, b_2, \ldots, b_k\}\}, \{\{a'_1, a'_2, \ldots a'_k\}, \{b'_1, b'_2, \ldots, b'_k\}\} \in \Gamma(2k, k, i)/\sigma$. Then the permutation $\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_k & b_1 & b_2 & \cdots & b_k \\ a'_1 & a'_2 & \cdots & a'_k & b'_1 & b'_2 & \cdots & b'_k \end{pmatrix}$ takes $\{\{a_1, a_2, \ldots, a_k\}, \{b_1, b_2, \ldots, b_k\}\}$ to $\{\{a'_1, a'_2, \ldots, a'_k\}, \{b'_1, b'_2, \ldots, b'_k\}\}$ and hence S_{2k} acts transitively on $\Gamma(2k, k, i)/\sigma$, $i = 1, k-1$. □

We now determine the automorphism group of $\Gamma(2k, k, i)/\sigma$, $i = 1, k-1$.

Theorem 5. $\mathrm{Aut}(\Gamma(2k, k, i)/\sigma) \cong S_{2k}$.

Proof. Let $X = \{\{1, 2, \ldots, k\}, \{k+1, k+2, \ldots, 2k\}\}$. Since the graph $\Gamma(2k, k, i)/\sigma$ is vertex transitive, to determine the order of the group, by the orbit–stabiliser theorem, it is enough to work out the order of the stabiliser of X, $\mathrm{Stab}(X)$.

Set $A = \{1, 2, \ldots, k\}$, $B = \{k+1, k+2, \ldots, 2k\}$. Let $i \in A$ and $j \in B$. Then the set C_i of vertices of the form $C_i = \{\{i\} \cup (B \setminus \{j\}), (A \setminus \{i\}) \cup \{j\}\}$ defines a distinct clique for each fixed $i \in A$ in $\Gamma(2k, k, i)/\sigma$. Hence there is a 1-1 correspondence between the set of cliques $C = \{C_i : i = 1, 2, \ldots, k\}$ and A.

Now any automorphism in $\mathrm{Stab}(X)$ that fixes $C = \{C_i : i = 1, 2, \ldots, k\}$ induces a permutation of S_A.

Similarly the set D_j of k vertices of the form $D_j = \{\{j\} \cup (A \setminus \{i\}), (B \setminus \{j\}) \cup \{i\}\}$ defines a distinct clique for each fixed $j \in B$ in $\Gamma(2k, k, i)/\sigma$. By similar argument any automorphism that fixes $D = \{D_i : i = k+1, k+2, \ldots, 2k\}$ induces as well a permutation of S_B.

It is easy to see that any automorphism in $\mathrm{Stab}(X)$ must either fix cliques in C or interchange them with those in D. Hence $|\mathrm{Stab}(X)| = 2(k!)(k!)$.

Now, since $(\Gamma(2k, k, i)/\sigma)$ has $\frac{(2k)!}{2(k)!(k)!}$ vertices then $|\mathrm{Aut}(\Gamma(2k, k, i)/\sigma)| = (2k)!$ So, in view of Lemma 2 $\mathrm{Aut}(\Gamma(2k, k, i)/\sigma) \cong S_{2k}$. □

Theorem 6. $\mathrm{Aut}(\Gamma(2k, k, i)) \cong S_{2k} \times S_2$, $i = 1, k-1$.

Proof. $\mathrm{Fix}(\Gamma(2k, k, i), f) \cong S_2$, $i = 1, k-1$ and by Theorem 4 the result follows. □

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