The analysis of indexed astronomical time series – X. Significance testing of $O - C$ data

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ABSTRACT

It is assumed that $O - C$ (‘observed minus calculated’) values of periodic variable stars are determined by three processes, namely measurement errors, random cycle-to-cycle jitter in the period, and possibly long-term changes in the mean period. By modelling the latter as a random walk, the covariances of all $O - C$ values can be calculated. The covariances can then be used to estimate unknown model parameters, and to choose between alternative models. Pseudo-residuals which could be used in model fit assessment are also defined. The theory is illustrated by four applications to spotted stars in eclipsing binaries.

Key words: methods: data analysis – methods: statistical – stars: variables: other.

1 INTRODUCTION

Koen (2005) recently presented a very brief review of the literature on rapid, non-evolutionary changes in the periods of variable stars. Examples included the effects of small abundance gradient changes on the the periods of pulsating stars, and of the ‘Applegate effect’ (e.g. Applegate 1992) on binary periods. Of course, convection is a well-known stochastic process, which has also been mentioned as a possible driver of small period changes in variable stars (e.g. Deasy & Wayman 1985; Wolff et al. 2002).

The type of period fluctuation described above could manifest itself in at least two different ways: as cycle-to-cycle ‘jitter’ in the period, or, if the effects are cumulative, as a systematic change in the period. The latter would not necessarily resemble the usually assumed sinusoids or low-order polynomials, but would rather be stochastic trends. This line of thinking was explored in some detail by Koen (1996); he found that good model fits to many $O - C$ (‘observed minus calculated’) data sets could be obtained with purely stochastic models.

The present paper is concerned with test statistics which can be used to decide which phenomena (measurement error, period jitter and/or systematic period changes) best explain a set of $O - C$ (‘observed minus calculated’) data sets. This contrasts with the typical approach in which regression techniques are used to try to ascertain the precise time of variation of the star.

The statistical model formulation is dealt with in Section 2; the test procedure is covered in Section 3; the model fit evaluation is discussed in Section 4; analysis of a few data sets is presented in Section 5; and conclusions are given in Section 6. Some of the necessary algebra is presented in the Appendix.

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This could further understanding of the levels of stability of periodic variability.

A very general model for the relatively smooth long-term variations in the period denoted by $\mu(t)$ is

$$\mu(t) = \mu_0 + \sum_{j=1}^{N} \xi_j,$$

where the $\xi_j$ are zero mean uncorrelated random variables with variance $\sigma^2_{\xi}$. Simulation examples of the efficacy of modelling smooth functions (typically resembling low-order polynomials) by the random walk model (1) can be found in Koen (1996); in a more general context see e.g. Harvey (1989).

The full model for the instantaneous period $P_j$ is then

$$P_j = \mu_j + \eta_j = \mu_0 + \sum_{k=1}^{j} \xi_k + \eta_j.$$  

Assume that there are $K + 2$ observations (timings) available, $t_0$, $t_1$, $t_2$, $\ldots$, $t_K, t_{K+1}$. Let the corresponding cumulative cycle numbers be $N_0 = 0$, $N_1$, $N_2$, $N_3$, $\ldots$, $N_{K+1} = N$, i.e. there are $n_j = N_j - N_{j-1}$ cycles between $t_j$ and $t_{j-1}$. The $j$th $O - C$ value is then

$$Z_j \equiv (O - C)_j = t_j - t_0 - N_j \overline{P}, \quad j = 1, 2, \ldots, K.$$  

In (3) $\overline{P}$ is the mean period over the $N = N_{K+1}$ cycles:

$$\overline{P} = \frac{t_{K+1} - t_0}{N}.$$  

Note that given the definition (3), $(O - C)_0 \equiv (O - C)_{K+1} = 0$.

The model specification is completed by noting that the $t_j$ are subject to zero mean measurement errors $e_j$ with constant variance $\sigma^2_e$:

$$t_j = T_j + e_j, \quad j = 0, 1, 2, \ldots, K + 1,$$

where the $T_j$ are the true, error-free values of the timings.

The expected values of all the $Z_j$ are zero, while the entries in their covariance matrix $\Sigma$ are derived in Appendix A.

Consider first the case where there is no long-term variation in the period, i.e. $\mu(t) = \mu_0$ in (1). Setting $\sigma_\xi = 0$ in (A6) and (A7) then leaves terms in $\sigma^2_{\xi}$ and $\sigma^2_e$ only. Examination of the two equations shows that these terms are of order unity and $N_j$ respectively. The implication is that the effects of period jitter, as modelled by the $\eta_j$, become comparable to those of measurement error in the $O - C$ observations for cumulative cycle numbers

$$N \gtrsim \left(\frac{\sigma_{\xi}}{\sigma_e}\right)^2.$$  

To illustrate, in the example in the second panel of Fig. 1, $\sigma_\xi/\sigma_e = 25$, hence the the effects of random intrinsic variations in the period could start making an impact for $N$ in excess of about 625.

By contrast, if the systematic period change can be modelled as in (1), its long-term effects grow much more rapidly ($\propto N^3$) with increasing cycle count.

3 Testing

Within the framework of the model formulated in Section 2, there are four possibilities:

- **M1**: the mean period is constant and there are no random cycle-to-cycle period variations ($\sigma_\xi = 0, \sigma_\eta = 0$);
- **M2**: the mean period is constant but there are random cycle-to-cycle period variations ($\sigma_\xi \neq 0, \sigma_\eta = 0$);
- **M3**: the mean period is variable but there are no random cycle-to-cycle period variations ($\sigma_\xi = 0, \sigma_\eta \neq 0$);
- **M4**: the mean period is variable and there are random cycle-to-cycle period variations ($\sigma_\xi \neq 0, \sigma_\eta \neq 0$).

One way of proceeding is to compare the four models by pairwise hypothesis testing. A possible test statistic is then the standard likelihood ratio

$$\Lambda_{ij} = 2(L_i - L_j),$$

where the Gaussian log likelihood is given by

$$L = -\frac{1}{2} \left[ K \log 2\pi + \log |\Sigma| + Z^\top \Sigma^{-1} Z \right].$$

In (8), $Z$ is the vector consisting of the $K$ observed $O - C$ values, and $|\Sigma|$ is the determinant of the covariance matrix $\Sigma$. The likelihoods $L_i$ and $L_j$ in (7) are calculated by setting the appropriate variances $(\sigma^2_\xi$ and/or $\sigma^2_\eta$) in $\Sigma$ equal to zero, and then maximizing with respect to $\sigma^2_e$ and the remaining variances (if any).

Aside from the erosion of significance levels due to multiple hypothesis testing, the $\Lambda_{ij}$ have non-standard distributions due to the fact that the hypotheses involve parameter values that lie on the boundaries of their domains (e.g. Andrews 2001). These and other complications are easily avoided by resorting to the use of model selection statistics: for our purposes useful forms are the Akaike and Bayes information criteria, defined by

$$AIC = -2 \log L + 2p + \frac{2p(p + 1)}{K - p - 1},$$

$$BIC = -2 \log L + p \log K$$

(e.g. Harvey 1989; Burnham & Anderson 2004). The symbol $p$ denotes the number of model parameters ($p = 1, 2, 3$ for models M1–M4 respectively). The two criteria essentially contrast a measure of how well the models fit the data (the likelihood term) with a penalty term for the number of fitted parameters ($2p$ and $p \log K$ respectively). The last term in the expression for the $AIC$ is a small
sample bias correction (e.g. Burnham & Anderson 2004). The maximum value of the likelihood is calculated for each of the four models, the results substituted into (9), and the model giving the minimum value of the information criteria selected.

Let AIC(i) be the Akaike information criterion for model Mi, and AIC\text{min} the minimum value of AIC(i) (i = 1, 2, 3, 4). Then

\[ p_{\lambda}(i) = \frac{\exp\left[-(AIC(i) - AIC_{\text{min}})/2\right]}{\sum_{j=1}^{k} \exp\left[-(AIC(j) - AIC_{\text{min}})/2\right]} \] (10)


\[ \text{can be interpreted as the probability that model } i \text{ is appropriate} \] (Burnham & Anderson 2004). The case of the Bayes information criterion is analogous: probabilities based on the BIC(i) will be denoted by \( p_B(i) \).

**4 FIT EVALUATION**

Once the optimal model has been selected, it is possible to investigate its fit to the data, similarly to the evaluation of regression residuals. The covariance matrix of the \( O - C \) values is

\[ \text{cov}(Z, Z) = \text{EZ}Z' = \Sigma \] (11)

(E is the expectation operator). The covariance matrix \( \Sigma \) is positive-definite and symmetric, and can therefore be written as

\[ \Sigma = LL' \] (12)

where \( L \) is lower triangular. This ‘Cholesky decomposition’ is discussed in, for example, Healy (1986). Defining

\[ u = L^{-1}Z, \] (13)

\[ \text{it follows from (11)–(13) that} \]

\[ S_a = \text{cov}(u, u) = E\left\{L^{-1}ZZ'[L^{-1}]^t\right\} \]

\[ = L^{-1}EZZ'[L^{-1}]^t \]

\[ = L^{-1}\Sigma[L^{-1}]^t \]

\[ = L^{-1}[LL'][L^{-1}]^{-1} \]

\[ = I. \] (14)

where \( I \) is the identity matrix.

A critical point is that (11), and hence all the equations following it, only applies if the model is appropriate: covariance matrices calculated on the basis of incorrect models will not reflect the covariances of the components of \( Z \). The implication is that the covariance matrix \( S_a \) of \( u \) is an identity matrix only if \( \Sigma \) is calculated for the correct model. Clearly, the components of \( u \) can be interpreted as model residuals; these have zero mean and unit variance, and are uncorrelated only if they are associated with the correct model. The \( u_i \) will be referred to as ‘pseudo-residuals’ in what follows, since they are not derived from a detailed model fit as in e.g. Koen (1996).

Of course, if none of M1–M4 is appropriate, then in general \( S_a \neq I \).

A few fairly elementary procedures can be applied to the pseudo-residuals in order to see whether these are indeed approximately identically distributed and uncorrelated. A simple plot of \( u_j \) against \( j \) is often sufficient; it can be supplemented by the autocorrelation function (acf) \( r(k) \) of the \( u_j \):

\[ r(k) = \frac{1}{K} \sum_{j=1}^{K-k} u_ju_{j+k}, \quad k = 1, 2, \ldots, J, \] (15)

usually calculated at lags \( k \) up to \( J = 10 \) or 15. A rough 5 per cent significance level for individual values of \( |r(k)| \) is \( 2/\sqrt{K} \). An overall significance level for the entire acf can be found from the portmanteau statistic

\[ Q = K \sum_{k=1}^{J} |r(k)|^2 \] (16)

which is approximately chi-squared distributed with \( J - p \) degrees of freedom, where \( p \) is the number of model parameters (see e.g. Chatfield 2004).

**5 SOME RESULTS**

As examples, the four data sets discussed by Sowell et al. (2001) are re-analysed. The timings are eclipse minima of four spotted stars in binary systems.

(i) **UV Psc**. The \( O - C \) diagram of the photoelectric data is plotted in Fig. 2, using \( P = 0.86104753 \) d. The results of fitting the four models, with and without the outlier visible in Fig. 2, are given in Table 1. The best model, according to both AIC and BIC, is M2 (period jitter only – no systematic change in period).

The solutions for \( \sigma_\varepsilon \) in M4 are so small that the model essentially reduces to M2. By implication M4 should be properly be excluded from the calculation of the \( p_\lambda \) and \( p_B \), making M2 even more dominant.

The pseudo-residuals are plotted in Fig. 3; all three sets are statistically uncorrelated. (Due to the fact that \( \sigma_\varepsilon = 0 \) in M4, its pseudo-residuals are very close to those of M2, and are therefore not plotted.)
Figure 3. The pseudo-residuals of the UV Psc data, after fitting models M1 (top panel), M2 (middle panel) and M3 (bottom panel). The model M4 pseudo-residuals are indistinguishable from those of the M2 model.

Figure 4. The $O - C$ diagram based on all observations of YY Gem, as tabulated by Sowell et al. (2001). The scale on the vertical axis is in units of 0.001 d.

Table 2. Results of fitting each of the four models to the YY Gem data. The estimated standard deviations are given in units of days. The last two columns are the Akaike and Bayes model probabilities [see equation (10)].

<table>
<thead>
<tr>
<th>Model</th>
<th>$\sigma_e$</th>
<th>$\sigma_\eta$</th>
<th>$\sigma_\xi$</th>
<th>$p_A$</th>
<th>$p_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>$2.5 \times 10^{-3}$</td>
<td>0</td>
<td>0</td>
<td>0.04</td>
<td>0.05</td>
</tr>
<tr>
<td>M2</td>
<td>$1.6 \times 10^{-4}$</td>
<td>$4.4 \times 10^{-5}$</td>
<td>0</td>
<td>0.39</td>
<td>0.39</td>
</tr>
<tr>
<td>M3</td>
<td>$1.8 \times 10^{-3}$</td>
<td>$5.3 \times 10^{-9}$</td>
<td>0</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>M4</td>
<td>$1.6 \times 10^{-4}$</td>
<td>$2.9 \times 10^{-5}$</td>
<td>$4.8 \times 10^{-9}$</td>
<td>0.12</td>
<td>0.12</td>
</tr>
</tbody>
</table>

(ii) YY Gem. The $O - C$ plot, based on $P = 0.814 282 20$ d, is shown in Fig. 4, and the model-fitting results are summarized in Table 2. There is not much to choose between M2 and M3, but M4 is less likely. There are only $N = 20$ data points, so the lack of a clear choice is perhaps not surprising.

The pseudo-residuals of all the models are consistent with being white noise.

(iii) CG Cyg. A period of 0.631 143 12 d was used to calculate the $O - C$ values in Fig. 5. Properties of the fitted models are summarized in Table 3: the model with both a systematic period change and intrinsic period scatter is preferred. The pseudo-residuals for models M2–M4 are plotted in Fig. 6, and their acfs are shown in Fig. 7. Somewhat surprisingly, the level of residual autocorrelation in M4 is higher than in M2 and M3: the portmanteau $Q$ statistics are 16.7, 10.1 and 21.6 for M2, M3 and M4 respectively [using $J = 10$ lags in equation (16)]. The $Q$-value for M4 is highly significant, meaning that there is considerable autocorrelation in the residuals. The result is due to the presence of the large residuals with alternating signs visible in the central section of the residual time series.

(iv) XY UMa. Fig. 8 is the $O - C$ diagram, calculated using $P = 0.478 995 01$ d. The model selection results in Table 4 are similar to those for UV Psc given in Table 1: M4 is essentially equivalent to

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Figure 5. The $O - C$ diagram based on the photoelectric photometry of CG Cyg, as tabulated by Sowell et al. (2001). The scale on the vertical axis is in units of 0.001 d.

Table 3. Results of fitting each of the four models to the CG Cyg data. The estimated standard deviations are given in units of days. The last two columns are the Akaike and Bayes model probabilities [see equation (10)].

<table>
<thead>
<tr>
<th>Model</th>
<th>$\sigma_e$</th>
<th>$\sigma_\eta$</th>
<th>$\sigma_\xi$</th>
<th>$p_A$</th>
<th>$p_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>$2.9 \times 10^{-3}$</td>
<td>0</td>
<td>0</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>M2</td>
<td>$2.1 \times 10^{-4}$</td>
<td>$5.2 \times 10^{-5}$</td>
<td>0</td>
<td>0.11</td>
<td>0.14</td>
</tr>
<tr>
<td>M3</td>
<td>$4.7 \times 10^{-4}$</td>
<td>0</td>
<td>$2.2 \times 10^{-8}$</td>
<td>0.13</td>
<td>0.18</td>
</tr>
<tr>
<td>M4</td>
<td>$2.7 \times 10^{-4}$</td>
<td>$3.1 \times 10^{-5}$</td>
<td>$1.9 \times 10^{-8}$</td>
<td>0.76</td>
<td>0.67</td>
</tr>
</tbody>
</table>

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Figure 6. The pseudo-residuals of the CG Cyg data, after fitting models M1–M4 (from top to bottom).
show the presence of random period fluctuations, superimposed on a systematic period change. The UV Psc and CG Cyg data also indicate significant.

More likely explanation of the results in this paper it is found that M2 (intrinsic period jitter) is a much more detailed, in that models 2 and 4 are also entertained. Indeed, XY UMa were subject to changes. The results in Tables 1–4 are in UV Psc and YY Gem, whereas the periods of both CG Cyg and YY Gem it is not possible to choose unambiguously between M2 and M3.

6 CONCLUSIONS

A brief summary of the steps in an analysis follows.

1. Determine the mean period as in (4). Using this value of \( \bar{P} \), find the cumulative cycle numbers \( N_i \) and hence the \( Z_j = (O - C)_j \) values [as in (3)].

2. For each of the models M1–M4, the log likelihood function is maximized with respect to its unknown parameters. The entries in the appropriate covariance matrix \( \Sigma \) are calculated using the formulae (A6) and (A7).

3. The information criteria \( AIC \) and \( BIC \) follow from (9), and the corresponding probabilities from (10) [and an analogous equation for the \( p_B(i) \)].

4. Calculate pseudo-residuals as in (13), where \( L \) is defined in (12). Plot these.

5. The residuals can be further assessed using (15) and (16).

Equation (8) is based on the assumption that the \( O - C \) values have a joint normal distribution. Provided that the measurement errors \( e_i \) are Gaussian, the assumption can be justified by referring to equation (A4) in the appendix:

\[
Z_j = e_j + \left( \frac{N_j}{N} - 1 \right) e_0 - \frac{N_j}{N} e_{K+1}
+ \sum_{k=1}^{N_j} \left( \eta_k + \sum_{i=1}^{k} \xi_i \right) - \frac{N_j}{N} \sum_{k=1}^{N} \left( \eta_k + \sum_{i=1}^{k} \xi_i \right)
\] (17)

For reasonably large cumulative cycle numbers \( N_j \), the sums over terms in \( \eta_k \) and \( \xi_i \) will each be normally distributed, by the central limit theorem (e.g. Mood, Graybill & Boes 1974). Therefore, provided that the \( e_i \) are Gaussian, \( Z_j \) in (17) is a sum over normally distributed variates, which is again Gaussian (e.g. Mood et al. 1974).

ACKNOWLEDGMENTS

Fruitful discussions with Professor Fred Lombard (University of Johannesburg) are gratefully acknowledged.

REFERENCES


Figure 7. The autocorrelation functions of the pseudo-residuals of models M2–M4 from Fig. 6. The broken lines are the approximate 95 per cent confidence bounds for zero correlation [see the text after equation (15)].

Figure 8. The \( O - C \) diagram of the photoelectric photometry of XY UMa, as tabulated by Sowell et al. (2001). The scale on the vertical axis is in units of 0.001 d.

Table 4. Results of fitting each of the four models to the XY UMa data. The estimated standard deviations are given in units of days. The last two columns are the Akaike and Bayes model probabilities [see equation (10)].

<table>
<thead>
<tr>
<th>Model</th>
<th>( \sigma_e )</th>
<th>( \sigma_\eta )</th>
<th>( \sigma_\xi )</th>
<th>( p_A )</th>
<th>( p_B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>4.4 \times 10^{-3}</td>
<td>0</td>
<td>0</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>M2</td>
<td>6.6 \times 10^{-4}</td>
<td>1.5 \times 10^{-4}</td>
<td>0</td>
<td>0.75</td>
<td>0.88</td>
</tr>
<tr>
<td>M3</td>
<td>1.7 \times 10^{-3}</td>
<td>0</td>
<td>4.5 \times 10^{-8}</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>M4</td>
<td>6.6 \times 10^{-4}</td>
<td>1.5 \times 10^{-4}</td>
<td>4.0 \times 10^{-15}</td>
<td>0.25</td>
<td>0.12</td>
</tr>
</tbody>
</table>

M2. The acfs of the pseudo-residuals of M2 and M4 are consistent with white noise, while the \( Q \)-statistic for model M3 is marginally significant.

Sowell et al. (2001) concluded that there are no period changes in UV Psc and YY Gem, whereas the periods of both CG Cyg and XY UMa were subject to changes. The results in Tables 1–4 are more detailed, in that models 2 and 4 are also entertained. Indeed, in this paper it is found that M2 (intrinsic period jitter) is a much more likely explanation of the \( O - C \) diagram of XY UMa than M3 (a systematic period change). The UV Psc and CG Cyg data also show the presence of random period fluctuations, superimposed on
APPENDIX A: THE COVARIANCES OF THE $O - C$ VALUES

The $j$th $O - C$ value, denoted by $Z_j$, is

$$Z_j = t_j - (t_0 + N_j \mathcal{P}), \quad j = 1, 2, \ldots, K$$

[see equation (3)], where the mean period is given by (4). For the model of equations (1), (2) and (5) it follows that

$$t_j = T_j + e_j = T_0 + \sum_{k=1}^{N_j} P_k + e_j$$

$$= T_0 + \sum_{k=1}^{N_j} (\eta_k + \mu_k) + e_j$$

$$= T_0 + N_j \mu_0 + \sum_{k=1}^{N_j} \left( \eta_k + \sum_{i=1}^{k} \xi_i \right) + e_j.$$  \hfill (A2)

Therefore

$$\mathcal{P} = \frac{1}{N} \left[ \epsilon_{K+1} - \epsilon_0 + N \mu_0 + \sum_{k=1}^{N} \left( \eta_k + \sum_{i=1}^{k} \xi_i \right) \right].$$ \hfill (A3)

Substituting (A2) and (A3) into (A1),

$$Z_j = e_j - \epsilon_0 + N_j \mu_0 + \sum_{k=1}^{N_j} \left( \eta_k + \sum_{i=1}^{k} \xi_i \right) - \frac{N_j}{N} \left[ \epsilon_{K+1} - \epsilon_0 + N \mu_0 + \sum_{k=1}^{N} \left( \eta_k + \sum_{i=1}^{k} \xi_i \right) \right]$$

$$= e_j + \left( \frac{N_j}{N} - 1 \right) \epsilon_0 - \frac{N_j}{N} \epsilon_{K+1} + \sum_{k=1}^{N_j} \left( \eta_k + \sum_{i=1}^{k} \xi_i \right) - \frac{N_j}{N} \sum_{k=1}^{N} \left( \eta_k + \sum_{i=1}^{k} \xi_i \right).$$ \hfill (A4)

The expected value of all the $Z_j$ is zero. Evaluation of the covariances is straightforward, except for those involving terms in $U_k = \left( \sum_{j=1}^{N_k} \sum_{i=1}^{j} \xi_i \right) = \sum_{j=1}^{N_k} (N_k - j + 1) \xi_j,$

and these are dealt with in detail. Assuming $k \leq \ell$,

$$\text{cov}(U_k, U_j) = \text{cov} \left[ \sum_{j=1}^{N_k} (N_k - j + 1) \xi_j, \sum_{i=1}^{N_\ell} (N_\ell - i + 1) \xi_i \right]$$

$$= \text{cov} \left[ \sum_{j=1}^{N_k} (N_k - j + 1) \xi_j, \sum_{i=1}^{N_\ell} (N_\ell - i + 1) \xi_i \right]$$

$$= \sigma^2 \sum_{j=1}^{N_k} (N_k - j + 1)(N_\ell - j + 1),$$

which is easily shown to reduce to

$$\text{cov}(U_k, U_j) = \frac{1}{6} N_k (N_k + 1) \sigma^2 (3N_\ell - N_k + 1).$$ \hfill (A5)

It follows from (A4) and (A5) that

$$\text{var}(Z_j) = 2 \left[ \left( \frac{N_j}{N} \right)^2 - \frac{N_j}{N} + 1 \right] \sigma^2 + N_j \left( 1 - \frac{N_j}{N} \right) \sigma^2$$

$$+ \frac{N_j}{6} \left[ (N_j + 1)(2N_j + 1) - 2 \frac{N_j}{N} (N_j + 1)(3N - N_j + 1) + \frac{N_j}{N} (N + 1)(2N + 1) \right] \sigma^2.$$ \hfill (A6)
Similarly,\n\[
\text{cov}(Z_k, Z_\ell) = \left[ \left( 1 - \frac{N_k}{N} \right) \left( 1 - \frac{N_\ell}{N} \right) + \frac{N_k N_\ell}{N^2} \right] \sigma_i^2 + N_k \left( 1 - \frac{N_\ell}{N} \right) \sigma_0^2 
\]
\[
+ \frac{N_k}{6} \left[ (N_k + 1)(3N_\ell - N_k + 1) - \frac{N_\ell}{N} (N_k + 1)(3N_k - N_\ell + 1) - \frac{N_\ell}{N} (N_k + 1)(3N - N_\ell + 1) + \frac{N_\ell}{N} (N + 1)(2N + 1) \right] \sigma_\ell^2
\]
(A7)

for \( k < \ell \).

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