

Stability of an SEIR epidemic model with independent stochastic perturbations

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Abstract

For an epidemic model of the type mentioned, we prove a theorem on almost sure exponential stability of the disease-free equilibrium. For small values of the diffusion parameter, σ , we describe the stability of the disease free equilibrium point in terms of an appropriate analogue, R_σ , of the basic reproduction number R_0 of the deterministic special case. Whenever $\sigma > 0$ then $R_\sigma < R_0$. For small values of σ , the stability theorem guarantees almost sure exponential stability whenever $R_\sigma < 1$. We also discuss the effect of increasing σ .

1. Introduction

The SIR model of disease dynamics considers the population in question as being divided into three compartments, see for instance the books [1] of Anderson and May or [2] of Brauer et al. Such a model can be modified to accommodate special features of the disease, conditions in the population or interventions on the epidemic. A combination of such interventions has been studied in the paper [3] of Meng and Chen, for instance. SIR models efficiently handle cases for which the disease takes effect relatively quickly in a newly infected individual. In cases where there are significant periods of incubation of the pathogen in a newly found host, it becomes necessary to accommodate this period of incubation when modeling, possibly by means of a delay or by introduction of another class (usually denoted by E for *exposed*) to capture newly infected individuals in the latent or incubation phase. SEIR models have been studied very extensively. Tuberculosis and HIV are examples of infections that are commonly modeled by including such a latent class. Among others, the paper [4] of Guo and Li presents a model which provides for several levels of intensity of infection. There are also models with multiple stages for other diseases, such as for instance the HIV model in Ref. [5] of Bhunu et al.

Another addition in the modeling of population dynamics of diseases is the introduction of stochasticity into models. This can bring to light new insights. Many systems in nature reveal stochasticity in itself or are subjected to stochastic perturbations. The need may arise from time to time, to accommodate such random phenomena into the modeling. Various approaches have been taken towards

incorporating random phenomena of nature into epidemiological models. One such approach is to utilize stochastic differential equations, and in the paper [6] of Gray et al. such models are referred to as *sde models*. Also, many different disease types have been modeled using sde's. Modeling of HIV through sde's appears, for instance, in the papers [7] by Dalal et al., [8] by Ding et al., [9] by Yang et al. and [10] by Mukandavire et al. The papers [11] by Tornatore et al. and [12] of Jovanović and Krstić present sde models of vector-borne diseases.

An important aspect that is investigated for a disease is the long term prevalence of the disease or whether it will ultimately vanish from the population. In fact, these stability studies may well be the most active research theme in epidemiological modeling. There are various notions of stability for systems of sde's, see the book [13] of Mao. Stochastic perturbation is capable of improving stability, or to stabilize an otherwise unstable equilibrium solution of a system. This point is discussed in Ref. [13] for instance. It is also illustrated in the paper [6] for the case of disease-free equilibrium of an SIS model and in Ref. [14].

In this paper we study an sde model with *independent* perturbations for a disease of the SEIR type. This means that starting with a certain deterministic compartmental model, the differential equations are perturbed by mutually independent white noise terms. In particular, for this type of model, the total population size itself is perturbed directly by white noise. Examples of such models can be found in the works [15,16,10,17–19] for instance. An sde model of an SEIR disease is studied in the paper [17] of Yang et al. The papers [19] of Yuan et al. and [18] of Yang and Mao study stability of sde multi-group SEIR models. The model used in this paper can be obtained as special cases of models in Refs. [17,18]. We prove a theorem on almost sure exponential stability, which shows that the stochastic perturbation enhances stability of the disease-free equilibrium. Sample simulations are presented to illustrate the main theorem.

2. Preliminaries

We assume throughout the paper that we have a filtered probability space

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P})$, which is complete and right continuous.

Let us consider a k -dimensional Wiener process $B(t)$ defined on this filtered probability space. Suppose that we have a k -dimensional stochastic differential equation with initial value as below.

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), \quad \text{with } x(0) = x_0. \quad (2.1)$$

The solution is denoted by $x(t, x_0)$. Let us suppose that $f(t, 0) = g(t, 0) = 0$ for all $t \geq 0$, so that the zero of \mathbb{R}^k is an *equilibrium point* of Eq. (2.1).

There are different variations of the notion of stability for equilibrium points of sde's. We shall focus on one of them.

Definition 2.1 ([13]). The equilibrium $x = 0$ of the system (2.1) is said to be *almost surely exponentially stable* on a subset Δ of \mathbb{R}^k if for all $x_0 \in \Delta$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln |x(t, x_0)| < 0.$$

The differential for a function $V(t, x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^k)$ by the formula

$$\mathcal{L}V = \frac{\partial V}{\partial t} + f^{\text{trp}} \frac{\partial V}{\partial x} + \frac{1}{2} \text{Trc} \left[g^{\text{trp}} \frac{\partial^2 V}{\partial x^2} g \right], \quad (2.2)$$

where Trc means *trace* and trp denotes the *transpose* of a matrix.

We require the following elementary inequality in the proof of the main result.

Lemma 2.2. For any real numbers a_1, a_2, \dots, a_k we have:

$$\sum_{r=1}^k a_r^2 \geq \frac{1}{k} \left(\sum_{r=1}^k a_r \right)^2.$$

Proof. Let $a = \sum_{r=1}^k a_r$. The statement is clearly true when $a = 0$. Otherwise we solve the constrained optimization problem:

$$\text{Minimize } \sum_{r=1}^k x_r^2 \quad \text{subject to } \sum_{r=1}^k x_r = a.$$

The lemma follows readily.

The following lemma on sequences of positive real numbers is quoted from Ref. [14] where it was proved and applied. It fulfils a similar role in this paper.

Lemma 2.3 ([14, Lemma 2.2]). For $k \in \mathbb{N}$, let $X(t) = (X_1(t), X_2(t), \dots, X_k(t))$ be a bounded \mathbb{R}^k -valued function. Let $(t_{0,n})$ be any increasing unbounded sequence of positive real numbers. Then there is a family of sequences $(t_{l,n})$ such that for each $l \in \{1, 2, \dots, k\}$, $(t_{l,n})$ is a subsequence of $(t_{l-1,n})$ and the sequence $X_l(t_{l,n})$ converges to the largest limit point of the sequence $X_l(t_{l-1,n})$.

3. The SEIR model

When introducing stochastic perturbations into a compartmental model, one may wish to perturb a specific parameter, or some or all of the variables. So for example the model of Ref. [7, p. 39] has only two of the four state variables carrying stochastic perturbations while in the model of Ref. [17] every variable is perturbed. There are models in which the transmission rate from susceptible to infected is perturbed such as in the Refs. [6,16,20] and others. We propose a model in which stochasticity is directly imposed on all the classes other than the class of susceptibles. This could be motivated by the uncertainty and behavioral change of individuals subsequent to infection. The model we present is a special case of models in Refs. [17,18].

The symbols S (susceptible), E (exposed), I (infectious) and R (removed) denote the number of individuals in the different compartments. The total population size at time t is $N(t)$. Therefore $N(t) = S(t) + E(t) + I(t) + R(t)$. The mortality rates for the different classes are μ_0 , μ_1 , μ_2 and μ_3 . The S -class experiences an inflow at a constant rate $\mu_0 K$. The effective rate of contact between infectives and susceptibles is β . The symbol u denotes the transmission rate from the latent class to the class of infectives and v is the recovery rate. The model parameters are all assumed to be positive constants. The dynamics of the disease in the population is given by the following system of sde's. The symbol $W(t) = (W_1(t), W_2(t), W_3(t))$ denotes a 3-dimensional Wiener process. The non-negative constants σ_1 , σ_2 and σ_3 denote the intensities of the stochastic perturbations.

3.1. The sde system

$$\begin{aligned} dS &= [\mu_0(K - S) - \beta SI]dt \\ dE &= [\beta SI - (u + \mu_1)E]dt + \sqrt{6}\sigma_1 E dW_1 \\ dI &= [uE - (v + \mu_2)I]dt + \sqrt{6}\sigma_2 I dW_2 \\ dR &= [vI - \mu_3 R]dt + \sqrt{6}\sigma_3 R dW_3. \end{aligned}$$

We assume that throughout the paper that

$$\mu_0 \leq \min\{\mu_1, \mu_2, \mu_3\}. \quad (3.1)$$

Let us write

$$\sigma = \min\{\sigma_1, \sigma_2, \sigma_3\}. \quad (3.2)$$

3.2. An analogue of the basic reproduction number

The following invariant $\mathcal{R}(\sigma, -)$ fulfils a central role in the main theorem on stability, similar to the role of the basic reproduction number in a deterministic model. To introduce a new parameter, a constant c , which is not a model parameter. Thus, for $0 < c \leq 1$, the number is defined as:

$$\mathcal{R}_{(\sigma, c)} = \frac{u\beta K}{(\mu_1 + u + c\sigma^2)(\mu_2 + v + c\sigma^2)}.$$

We note that in the deterministic case of the model system (3.1), i.e., when $\sigma_1 = \sigma_2 = \sigma_3 = 0$, then the aforementioned invariant coincides with the basic reproduction number R_0 of the disease, as calculated in Ref. [4]. We shall write $R(\sigma,1)$ simply as R_σ .

3.3. Positivity of solutions

In order for our model to be realistic, solutions will have to be non-negative. In our quest for feasible solutions we shall refer to the set Δ , which is defined as follows.

$$\Delta = \{x \in \mathbb{R}^4 \mid x_1 > 0, x_2 > 0, x_3 > 0, x_4 > 0\}.$$

The existence of positive solutions can be deduced from Ref. [17, Theorem 3.1] or Ref. [18, Theorem 3.1]. Thus, for each $(S_0, E_0, I_0, R_0) \in \Delta$, sde system 3.1 admits a unique solution $(S(t), E(t), I(t), R(t))$ on $t \geq 0$, and this solution remains in Δ almost surely.

In the models of Refs. [17,18], the non-zero values for the diffusion parameters associated with the S -classes, complicate the convergence possibilities. Since in our model we have no perturbation to S , convergence of S becomes much more tractable. The following subset Ω_1 of sample paths will be of interest.

$$\Omega_1 = \{w \in \Omega \mid S(t, w(t)), E(t, w(t)), I(t, w(t)), R(t, w(t)) \in \Delta \text{ for } t \geq 0\}.$$

Ref. [18] it follows that $P(\Omega \setminus \Omega_1) = 0$. In the remainder of this paper we assume that sample paths are restricted to Ω_1 .

Proposition 3.4. *If $(S_0, E_0, I_0, R_0) \in \Delta$, then almost surely, $S(t) \leq K$ for all $t > 0$.*

Proof. Given any path (in Ω_1), then

$$\frac{d(S - K)}{dt} = -\mu_0(S - K) - \beta SI \leq -\mu_0(S - K).$$

Therefore $S(0) < K$ implies that $S(t) < K$ for all $t > 0$.

In the following remarks we present some notation and observations that are important for the proof of the main theorem.

Remark 3.5. Let us fix any positive real numbers a_0, a_1, a_2, a_3 . We define the following stochastic processes:

$$z(X(t)) = a_0(K - S(t)) + a_1E(t) + a_2I(t) + a_3R(t),$$

$$Y_E(X(t)) = \frac{E(t)}{z(X(t))}$$

and similarly

$$Y_I = z^{-1}I, \quad Y_R = z^{-1}R, \quad Y_Q = z^{-1}(K - S).$$

Note that for every $t > 0$ we have:

$$0 < Y_E(t) \leq \frac{1}{a_1}, \quad 0 < Y_I(t) \leq \frac{1}{a_2}, \quad 0 < Y_R(t) \leq \frac{1}{a_3}, \quad 0 < Y_Q(t) \leq \frac{1}{a_0}.$$

In particular, the stochastic processes are bounded above by $\max\{\frac{1}{a_0}, \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}\}$, and are non-negative. Also we have the following identity:

$$\text{for every } t > 0, \quad a_0Y_Q(t) + a_1Y_E(t) + a_2Y_I(t) + a_3Y_R(t) = 1. \quad (3.3)$$

Remark 3.6. The following stochastic processes will be encountered in the proof of the main theorem. Let

$$M(t) = \sqrt{6} \sum_{l=1}^3 M_l(t)$$

with

$$M_1(t) = \int_0^t \frac{\sigma_1 E(y)}{z(X(y))} dW_1(y), \quad M_2(t) = \int_0^t \frac{\sigma_2 I(y)}{z(X(y))} dW_2(y), \quad M_3(t) = \int_0^t \frac{\sigma_3 R(y)}{z(X(y))} dW_3(y).$$

Regarding the quadratic variations of the stochastic integral $M_l(t)$ we have

$$\int_0^t \left(\frac{\sigma_1 E(y)}{z(X(y))} \right)^2 dy \leq \frac{\sigma_1^2 t}{a_1^2}, \quad \int_0^t \left(\frac{\sigma_2 I(y)}{z(X(y))} \right)^2 dy \leq \frac{\sigma_2^2 t}{a_2^2}, \quad \int_0^t \left(\frac{\sigma_3 R(y)}{z(X(y))} \right)^2 dy \leq \frac{\sigma_3^2 t}{a_3^2}.$$

We now apply the strong law of large numbers for local martingales, as in Ref. [13, p. 12] for instance. We can deduce that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{l=1}^3 M_l(t) = 0 \quad (\text{a.s.}).$$

4. Stability theorems

Stability results of sde models for bigger values as well as smaller values of the perturbation parameters appear, for instance, in Ref. [20, Theorem 5], Ref. [7, p. 52] and in Ref. [14]. In the main theorem, **Theorem 4.1**, we investigate stability for mainly the smaller values of the perturbation parameters. Recall that $\sigma = \min\{\sigma_j \mid j = 1, 2, 3\}$.

Theorem 4.1. Suppose that c is a constant with $0 < c \leq 1$ and such that the following inequality holds:

$$c\sigma^2 \leq \mu_0. \quad (4.1)$$

If $R_{\sigma,c} < 1$, then the disease-free equilibrium is almost surely exponentially stable.

Proof. Let us assume the conditions of the theorem to hold. The condition $R_{\sigma,c} < 1$ is equivalent to the following inequality:

$$\frac{u}{\mu_1 + u + c\sigma^2} \beta K - (\mu_2 + v + c\sigma^2) < 0.$$

We can choose a number $a_0 > 0$ sufficiently small in order that the following inequality holds:

$$\left(a_0 + \frac{u + a_0}{\mu_1 + u + c\sigma^2} \right) \beta K - (\mu_2 + v + c\sigma^2) < 0.$$

Now we fix the following numbers. Let

$$a_1 = \frac{u + a_0}{\mu_1 + u + c\sigma^2} \quad \text{and} \quad \text{let } a_2 = 1.$$

We can choose a positive number a_3 sufficiently small, such as to satisfy the inequality:

$$\left(a_0 + \frac{u + a_0}{\mu_1 + u + c\sigma^2} \right) \beta K - (\mu_2 + v + c\sigma^2) + a_3 v < 0. \quad (4.2)$$

Let us write $(S(t), E(t), I(t), R(t)) = X(t)$. With the numbers a_i as declared above, we define a stochastic process $z(X(t))$ as in [remark 3.5](#). Since $z(X(t)) > 0$ for all $t > 0$ (recall that we assume the sample paths to be in Ω_1), we can also define $X(t)$.

$$z(X(t)) = a_0(K - S(t)) + a_1 E(t) + a_2 I(t) + a_3 R(t), \quad V(X(t)) = \ln z(X(t)).$$

In order to prove our theorem it suffices to prove that $z(X(t))$ converges exponentially to zero (a.s.). Using Itô's formula (see Ref. [13] for instance) we can express the stochastic process $V(X(t))$ as:

$$V(X(t)) = V(X(0)) + \int_0^t \mathcal{L}V(X(u))du + M(t),$$

with $M(t)$ as in Remark 3.6, where we noted that

$$\lim_{t \rightarrow \infty} \frac{1}{t} M(t) = 0 \quad (\text{a.s.}).$$

Therefore

$$\limsup_{t \rightarrow \infty} \frac{1}{t} V(X(t)) = \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathcal{L}V(X(y))dy \quad (\text{a.s.}),$$

and it follows that we only need to prove that

$$\limsup_{t \rightarrow \infty} \mathcal{L}V(X(t)) < 0 \quad \text{a.s.}$$

We write down the expression for $\mathcal{L}V(X(t))$, given the system of stochastic differential equations of the SEIR model.

$$\begin{aligned} \mathcal{L}V(X) = & \frac{1}{z} [-a_0(\mu_0(K - S) - \beta SI) + a_1(\beta SI - (u + \mu_1)E) + a_2(uE - (v + \mu_2)I) + a_3(vI - \mu_3R)] \\ & - \frac{3}{z^2} ((a_1\sigma_1E)^2 + (a_2\sigma_2I)^2 + (a_3\sigma_3R)^2). \end{aligned} \quad (4.3)$$

For every sample path w of the 3-dimensional Wiener process $W(t)$, there exists an unbounded increasing sequence (τ_n^w) of positive time values for which

$$\lim_{n \rightarrow \infty} \mathcal{L}V(X(\tau_n, w(\tau_n))) = \limsup_{t \rightarrow \infty} \mathcal{L}V(X(t, w(t))).$$

Fix such a sequence. Then by Lemma 2.3 there exists a subsequence (τ_n^w) for which the following limit exists (see notation in Remark 3.6):

$$\lim_{n \rightarrow \infty} (S(t_n^w, w(t_n^w)), Y_E(X(t_n^w, w(t_n^w))), Y_I(X(t_n^w, w(t_n^w))), Y_R(X(t_n^w, w(t_n^w)))) .$$

This enables us to define the following limits (and now we suppress the path w in the notation):

$$s = \lim_{n \rightarrow \infty} S(t_n), \quad f = \lim_{n \rightarrow \infty} Y_E(X(t_n)), \quad i = \lim_{n \rightarrow \infty} Y_I(X(t_n)), \quad r = \lim_{n \rightarrow \infty} Y_R(X(t_n)),$$

$$q = \lim_{n \rightarrow \infty} Y_Q(X(t_n)).$$

From the identity (3.2) it follows that

$$a_0 q + a_1 f + a_2 i + a_3 r = 1. \quad (4.4)$$

Let us write

$$\Lambda = \lim_{n \rightarrow \infty} \mathcal{L}(\ln[z(t_n)]).$$

Then from the Eq. (4.3) it follows that

$$\Lambda = -a_0 \mu_0 q + a_0 \beta s i + a_1 (\beta s i - (u + \mu_1) f) + a_2 (u f - (v + \mu_2) i) + a_3 (v i - \mu_3 r) - 3((a_1 \sigma_1 f)^2 + (a_2 \sigma_2 i)^2 + (a_3 \sigma_3 r)^2). \quad (4.5)$$

In the sequence of inequalities which follows below, we use the fact that $c \leq 1$ implies $-\sigma^2 \leq -(c\sigma)^2$. At some point we use identity $a_1 f + a_2 i + a_3 r = 1 - a_0 q$ and the inequality $a_1 f + a_2 i + a_3 r \leq 1$, both of which follow from the identity (4.4).

$$\begin{aligned} -(a_1 \sigma_1 f)^2 - (a_2 \sigma_2 i)^2 - (a_3 \sigma_3 r)^2 &\leq -c\sigma^2[(a_1 f)^2 + (a_2 i)^2 + (a_3 r)^2] \\ &\leq -\frac{1}{3}c\sigma^2(a_1 f + a_2 i + a_3 r)^2 \quad (\text{from Lemma 2.2}) \\ &= -\frac{c\sigma^2}{3}(a_1 f + a_2 i + a_3 r)(1 - a_0 q) \\ &= -\frac{c\sigma^2}{3}(a_1 f + a_2 i + a_3 r) + \frac{c\sigma^2}{3}(a_1 f + a_2 i + a_3 r)a_0 q \\ &\leq -\frac{c\sigma^2}{3}(a_1 f + a_2 i + a_3 r) + \frac{c\sigma^2}{3}a_0 q \\ &\leq -\frac{c\sigma^2}{3}(a_1 f + a_2 i) + \frac{c\sigma^2}{3}a_0 q. \end{aligned}$$

Therefore we obtain the following inequality:

$$\Lambda \leq -a_0 \mu_0 q + a_0 \beta s i + a_1 (\beta s i - (u + \mu_1) f) + a_2 (u f - (v + \mu_2) i) + a_3 (v i - \mu_3 r) - c\sigma^2(a_1 f + a_2 i) + c\sigma^2 a_0 q. \quad (4.6)$$

From Proposition 3.4 it follows that $s \leq K$. Furthermore we observe that

$$-a_1(u + \mu_1 + c\sigma^2)f + a_2uf = -a_0f.$$

This yields

$$\Lambda \leq i[(a_0 + a_1)\beta K - (v + \mu_2) + a_3v] - a_0\mu_0q - a_0f - a_3\mu_3r - c\sigma^2a_2i + c\sigma^2a_0q.$$

Therefore we obtain an inequality

$$\Lambda \leq A_0q + A_1f + A_2i + A_3r$$

with

$$\begin{aligned} A_0 &= a_0 [c\sigma^2 - \mu_0], \\ A_1 &= -a_0, \\ A_2 &= \left(a_0 + \frac{u + a_0}{\mu_1 + u + c\sigma^2} \right) \beta K - (\mu_2 + v + c\sigma^2) + a_3v, \\ A_3 &= -a_3\mu_3. \end{aligned}$$

From the inequality (4.2) it follows that $A_2 < 0$. The coefficient A_0 is negative due to the condition (4.1). Therefore all the coefficients A_0 , A_1 , A_2 and A_3 are negative. Furthermore, $a_0i + a_1q + a_2r + a_3f = 1$ and thus the limits i, q, r, f cannot all be zero. Therefore $\Lambda < 0$, and the proof is complete.

Remark 4.2. The role of the constant c in Theorem 4.1 is as follows. If the requirements of Theorem 4.1 are fulfilled for a given value σ^* of σ , then by increasing the value of σ we will still have the disease-free equilibrium to be almost surely exponentially stable.

Theorem 4.3. *If $R_0 < 1$ then the model is almost surely exponentially stable.*

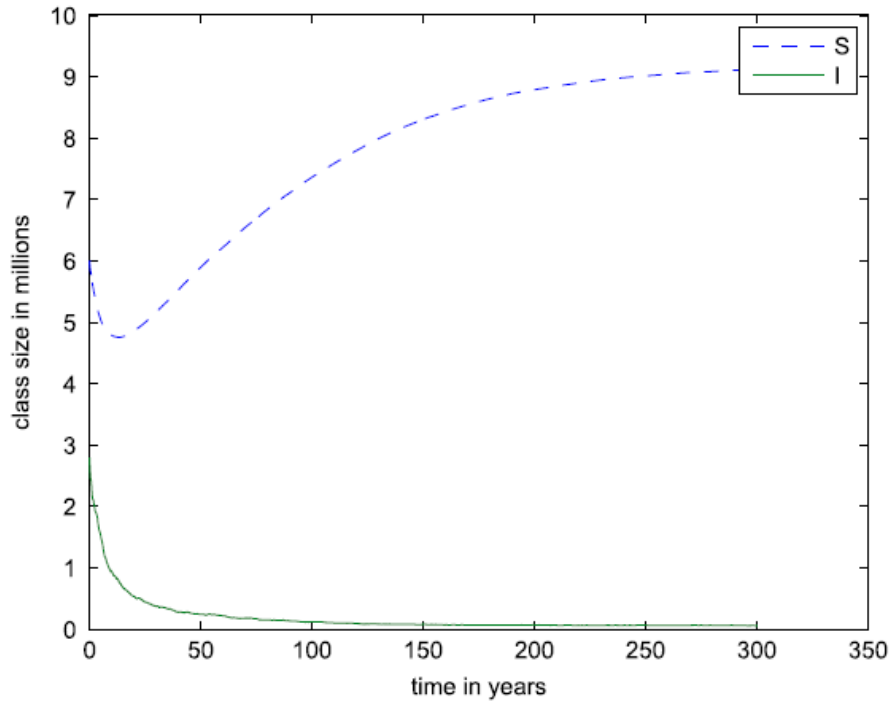


Fig. 1. For $\mathcal{R}_{(\sigma,1)} = 1.0510 > 1$, Theorem 4.1 does not guarantee almost sure exponential stability.

Proof. The proof follows very much along the argument of the proof of Theorem 4.1, but is much simpler in that we completely ignore the diffusion parameters. We choose $a_1 = (u + a_0)(\mu_1 + u)^{-1}$ and the Eq. (4.5) is simplified to get the inequality:

$$\Lambda \leq -a_0 \mu_0 q + a_0 \beta s i + a_1 (\beta s i - (u + \mu_1) f) + a_2 (u f - (v + \mu_2) i) + a_3 (v i - \mu_3 r).$$

The proof follows similarly as for Theorem 4.1.

5. Numerical examples

The graphs in Figs. 1–3 were simulated with the model parameters as follows:

$$K = 10 \text{ million}, \quad \beta = 0.024, \quad u = 0.05, \quad v = 0.15, \\ \mu_0 = 0.017, \quad \mu_1 = 0.0172, \quad \mu_2 = 0.0185, \quad \mu_3 = 0.0172,$$

and initial conditions (in units of millions)

$$S_0 = 6, \quad E_0 = 1, \quad I_0 = 2.8, \quad R_0 = 0.2.$$

The simulations run over different time horizons as indicated on the graphs. Now we note that these parameters give in the deterministic case (i.e., with $\sigma = 0$), a value of $R_0 = 1.0598$ to the basic reproduction number. Consequently the system eventually approaches an endemic equilibrium point with coordinates:

$$S^* = \frac{uv + u\mu_2 + v\mu_1 + \mu_1\mu_2}{u\beta},$$

$$E^* = \frac{\mu_0(uK\beta - uv - u\mu_2 - v\mu_1 - \mu_1\mu_2)}{u\beta(u + \mu_1)},$$

$$I^* = \frac{\mu_0(uK\beta - uv - u\mu_2 - v\mu_1 - \mu_1\mu_2)}{\beta(uv + u\mu_2 + v\mu_1 + \mu_1\mu_2)},$$

$$R^* = \frac{v\mu_0(uK\beta - uv - u\mu_2 - v\mu_1 - \mu_1\mu_2)}{\mu_3\beta(uv + u\mu_2 + v\mu_1 + \mu_1\mu_2)}.$$

The numerical values are:

$$S^* = 9.4360, \quad E^* = 0.1427, \quad I^* = 0.0423, \quad R^* = 0.3692.$$

The graphs show only the curves S and I , and we note (from the system of sde's) that $S'(t)$ is continuous.

For the simulation of Fig. 1, we choose $\sigma = 0.02$. Then we obtain $R(\sigma,1) = 1.0510$. A different value of c will not decrease the value of $R(\sigma,c)$. Therefore Theorem 4.1 does not guarantee almost sure exponential stability in this case.

In Fig. 2 we use $\sigma = 0.1$ and then we have $R(\sigma,1) = 0.8708 < 1$. With the requirements of Theorem 4.1 fulfilled, in this case the theorem asserts that the disease-free equilibrium is almost surely exponentially stable.

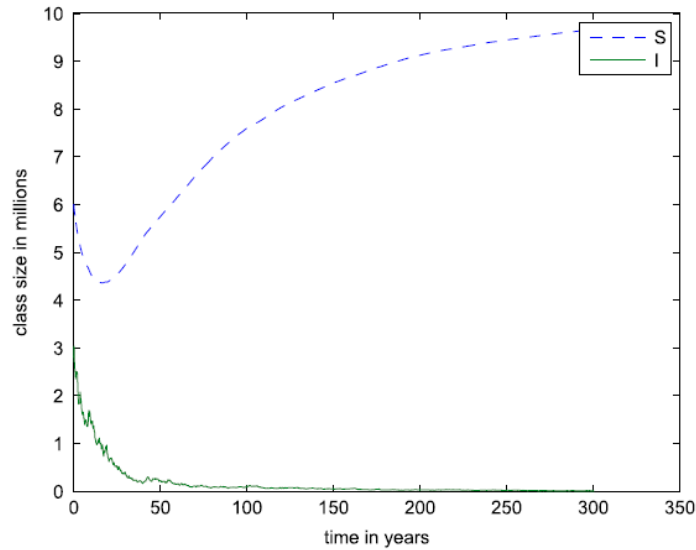


Fig. 2. Here we have $\mathcal{R}_{(\sigma,1)} = 0.8708 < 1$, and by Theorem 4.1 the disease-free equilibrium is almost surely exponentially stable.

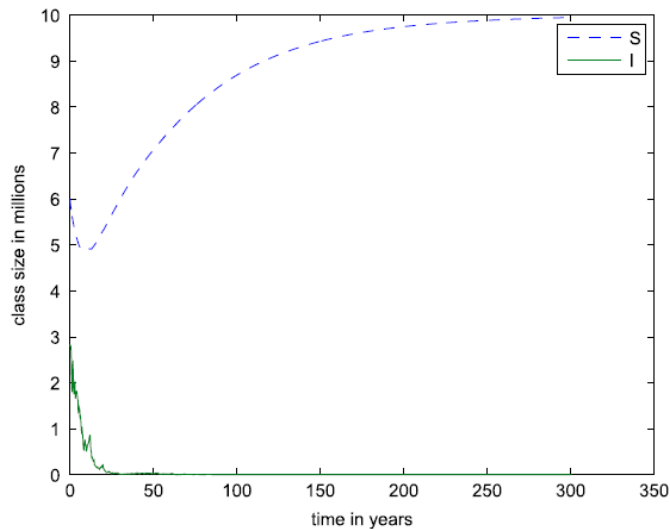


Fig. 3. With $\sigma = 0.25$, bigger than in Fig. 2, and the other parameters the same, we expect a more rapid convergence to the disease-free.

Having fulfilled the conditions of Theorem 4.1 with our previous choice of parameters and led by Remark 4.2, we choose $\sigma = 0.25$, and we expect the disease-free equilibrium to have almost sure exponential stability. In fact in this case, convergence to the disease-free equilibrium is expected to happen faster. Fig. 3 shows a simulation of the solution paths.

Finally, we show that the condition $\mathcal{R}_{(\sigma,c)} < 1$ is not sufficient to guarantee almost sure exponential stability of the disease-free equilibrium. In Fig. 4 we show a simulation with all the deterministic parameter values as before except that β is chosen at a higher value; $\beta = 0.04$. Taking $\sigma = 0.25$ gives $\mathcal{R}_{(\sigma,1)} = 0.6675 < 1$ (deterministic, $\mathcal{R}_0 = 1.7662$). For all the model parameters as chosen now, it is impossible to find a positive number c to fulfil the condition $c\sigma^2 \leq \mu_0$ of Theorem 4.1. Consequently the theorem does not guarantee the disease-free

equilibrium to be almost surely exponentially stable. Indeed, the graph in Fig. 4 does not show any tendency towards the disease-free equilibrium.

6. Conclusion

There are only a few research publications on sde modeling of SEIR dynamics. The paper [18] presents results on long time behavior of the system and in particular on convergence of S to its disease-free equilibrium value. It includes a study of the asymptotic probability distribution of the variables S_k . In the paper [17] is proved a theorem on the asymptotic value of the time average of the deviation from the disease-free equilibrium, subject to certain upper bounds on the perturbation parameters and with R_0 assumed to not exceed unity. The same paper gives results on limits of time averages, in particular, m value. Also, for the special case of S being unperturbed (as in our model), in Ref. [17] the system is shown to be stochastically stable in the large, see Ref. [17, Remark 3.1].

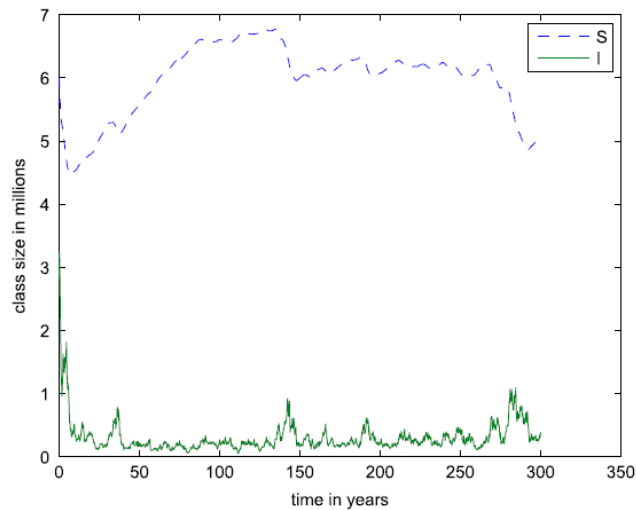


Fig. 4. In Theorem 4.1, the condition $\mathcal{R}_{(\sigma, c)} < 1$ is not sufficient to guarantee almost sure exponential stability of the disease-free equilibrium.

In relation to the theme of this paper, we particularly want to note the paper [6] which proves a stability result allowing for R_0 to exceed the value 1. The current paper makes a useful contribution to understanding the disease-free equilibrium of the given sde model of an SEIR disease. Our main result has an uncomplicated formulation and guarantees almost sure exponential stability of the disease-free equilibrium value for $R_\sigma < 1$. This implies an improvement on the condition $R_0 < 1$ required for local asymptotic stability in the deterministic case.

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