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Multiple regression of GRB luminosity on light-curve properties

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ABSTRACT

Statistical relations between the luminosity L of gamma-ray burster (GRBs) and several of their other observable properties have been discovered. Four of these properties are considered here: the spectral lag; a measure of the light-curve variability; peak energy and the minimum light-curve rise time. Data are taken from a tabulation by Schaefer. Log–log regression of L on various combinations of the four light-curve properties, as well as redshift, is considered, using conventional multiple linear regression, and multiple errors-in-variables regression. Several cases are found in which the regression coefficients of more than one luminosity indicators are significant. In particular, the simultaneous regression of luminosity on peak energy, spectral lag, minimum rise time and redshift is meaningful.

Key words: methods: statistical – gamma-rays: bursts.

1 INTRODUCTION

Many papers have been written on the prediction of gamma-ray burster (GRB) luminosities from their light-curve properties. Much of this work has been motivated by the desire to eventually produce reliable distance estimators for these objects, since their extreme luminosities make them observable over cosmic distances. This paper focuses on the relations between luminosity L and four of the proposed luminosity indicators. This particular choice was made because the data necessary for the study are conveniently available at a single source, namely Schaefer (2007).

The four luminosity indicators considered are as follows: the spectral lag $\tau_{\rm L}$ between hard and soft radiation (e.g. Norris, Marani & Bonnell 2000; Schaefer 2004; Tsutsui et al. 2008); the GRB peak energy $E_{\rm p}$ (e.g. Schaefer 2003b; Yonetoku et al. 2004; Firmani et al. 2006); the minimum light-curve rise time $\tau_{\rm R}$ (e.g. Schaefer 2007; Dado, Dar & De Rújula 2007) and a measure V of the 'spikiness' of the light curve (e.g. Reichart et al. 2001; Guidorzi et al. 2006; Rizzuto et al. 2007). Further references can also be found in these papers. The interested reader is referred to Schaefer (2007) for the exact definitions of the independent variables. Dar et al. (2007) provide a theoretical understanding, in terms of the 'cannonball' theory of GRBs, of the relation between luminosity L and the four selected indicators.

The paper is concerned with multivariate linear regression of L on various combinations of the independent variables $\tau_{\rm L}$, $E_{\rm p}$, $\tau_{\rm R}$ and V. Each regression further contains an intercept, and the independent variable $(1 + z)$. Logarithmic transformations (base 10) are made of all variables.

Noteworthy features of this paper are as follows.

(i) Including redshift explicitly as an independent variable in the regression. Motivation for this can be found in, for example, Lloyd-Ronning, Fryer & Ramirez-Ruiz (2003) and Tsutsui et al. (2008). As a concomitant, the four independent variables mentioned above are used in their observer-frame forms: they are not multiplied by powers of $(1 + z)$ in order to adjust them to the rest frame.

(ii) Regression of the luminosity on combinations of the four light-curve properties, instead of considering the luminosity indicators one at a time. Multiple regression is, of course, more efficient, since different independent variables may carry different information about the dependent variable. Table 1 gives correlations between the various luminosity indicators considered in this paper. The calculations are based on the $N = 28$ GRB in table 4 of Schaefer (2007) for which each of the five variables is available. Although few of the correlations are negligible (the two-sided 1 per cent significance limit is 0.46), the largest value of 0.73 suggests that the overlap in information between indicators in not excessive. The variables $E_{\rm p}$ and $(1 + z)$, in particular, are weakly correlated with the rest.

The point is further illustrated by the partial correlation coefficients with luminosity listed in Table 2. The correlation between L and parameter x was calculated by first removing the (linear) influence of the other four variables in the table separately from L and x, and then correlating the residuals. The data used are the same as for Table 1 (with the addition of luminosity); the null hypothesis of zero partial correlation can

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Table 1. Correlations between the luminosity indicators.

	$(1+z)$	$\tau_{\rm L}$	V	$E_{\rm p}$	$\tau_{\rm R}$
$(1+z)$	1.00	−0.24	0.16	−0.07	−0.29
$\tau_{\rm L}$		1.00	−0.65	−0.44	0.73
V			1.00	0.19	−0.61
$E_{\rm p}$				1.00	−0.25
$\tau_{\rm R}$					1.00

Table 2. Partial correlations of luminosity with $(1+z)$ and each of the four luminosity indicators. Significance levels are given in the second line of the table.

	$(1+z)$	$\tau_{\rm L}$	V	$E_{\rm p}$	$\tau_{\rm R}$
Partial correlation	0.85	−0.59	0.11	0.73	−0.32
Significance p	0	0.002	0.61	0	0.13

then be tested by reference to a Student's t distribution with $N-6=22$ degrees of freedom (e.g. Anderson 2003). Clearly $(1+z)$, $\tau_{\rm L}$ and $E_{\rm p}$ each contains unique information about L.

(iii) Taking account of the effect of measurement errors on the estimated regression slopes.

(iv) Using a non-parametric method, the jackknife, to estimate regression uncertainties.

It is fair to mention at this point that there has been a debate in the literature regarding the reality of the statistical relations found between luminosity and the various indicators (Butler et al. 2007, but see also Ghirlanda et al. 2008; Nava et al. 2008). Potential sources of spurious regression which have been mentioned include GRB selection effects.

The necessary statistical theory is summarized in the next section of the paper, and results are given in Section 3. Conclusions are given in Section 4.

Throughout the paper, an estimated parameter is considered significant (or 'meaningful') if it is at least two of its standard errors in size. This is a standard approach, and corresponds to a normal distribution two-sided significance level of 5 per cent. A fitted model is deemed acceptable (or 'satisfactory') if *all* of its parameters are significant.

2 STATISTICAL THEORY

The regression equation is

$$y_j = \beta_1 x_{1j} + \cdots + \beta_{Kj} x_{Kj} + q_j = \boldsymbol{x}_j \boldsymbol{\beta} + q_j \qquad j = 1, 2, \ldots, N, \tag{1}$$

where x_{kj} is the jth value of kth ($k = 1, 2, \ldots, K$) independent variable (luminosity indicator) and N is the number of GRB for which complete data are available. The residual error is given by q_j. The vectors $\boldsymbol{x}_j$ and β are

$$\boldsymbol{x}_j = [x_{1j} \quad x_{2j} \quad \ldots \quad x_{Kj}] \qquad \boldsymbol{\beta} = [\beta_1 \quad \ldots \quad \beta_{Kj}]',$$

where the prime indicates a transpose. The dependent variable $y = \log L$, $x_{1j} = 1$ (corresponding to the intercept β_1) and the remaining x_{kj} ($k = 2, 3, \ldots, K$) are (log) luminosity indicators.

The ordinary least-squares (OLS) estimate of β is given by

$$\boldsymbol{\beta}_{\rm LS} = \boldsymbol{M}_{xx}^{-1} \boldsymbol{M}_{xy}, \tag{2}$$

where

$$\boldsymbol{M}_{xx} = \sum_{j=1}^{N} \boldsymbol{x}_j' \boldsymbol{x}_j \qquad \boldsymbol{M}_{xy} = \sum_{j=1}^{N} \boldsymbol{x}_j' y_j. \tag{3}$$

The estimator $\boldsymbol{\beta}_{\rm LS}$ is known to be biased if the x_{kj} are subject to measurement errors. There is an extensive literature on bias-free alternatives to the estimator $\boldsymbol{\beta}_{\rm LS}$. The methodology described below is taken from Fuller (1987) to whom the interested reader is referred for details. Let

$$Y_j = y_j + v_j \qquad X_{kj} = x_{kj} + u_{kj},$$

i.e. Y_j and X_{kj} are the *observed* values (as opposed to unobservable true values) of the dependent and independent variables; v_j and u_{kj} are the measurement errors. It is assumed that the measurement error variances are known:

$$\mathrm{var}(v_j) = \sigma_{vj}^2 \qquad \mathrm{var}(u_{kj}) = \sigma_{u,kj}^2.$$

It is also assumed that the errors on different variables, and on different measurements on a given variable, are uncorrelated:

$$\mathrm{cov}(u_{kj}, u_{\ell i}) = 0 \qquad \text{unless} \quad k = \ell, \ j = i.$$

Also,

$$\mathrm{cov}(u_{kj}, v_i) = 0 \qquad \text{for all} \quad k, j, i.$$

The following definitions are useful.

$$\Sigma_{uj} = \begin{bmatrix} \sigma^2_{u,1j} & 0 & 0 & \dots & 0 \\ 0 & \sigma^2_{u,2j} & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ 0 & \dots & & & \sigma^2_{u,Kj} \end{bmatrix}$$

$$\Sigma_{ej} = \begin{bmatrix} \sigma^2_{vj} & 0 & 0 & 0 & \dots & 0 \\ 0 & \sigma^2_{u,1j} & 0 & 0 & \dots & 0 \\ 0 & 0 & \sigma^2_{u,2j} & 0 & \dots & 0 \\ \vdots & 0 & 0 & \ddots & & \vdots \\ 0 & \dots & & & & \sigma^2_{u,Kj} \end{bmatrix}$$

$$\boldsymbol{U} = \sum_{j=1}^{N} \Sigma_{uj} \quad \boldsymbol{E} = \sum_{j=1}^{N} \Sigma_{ej} \quad \boldsymbol{V} = \sum_{j=1}^{N} \begin{bmatrix} y_j \\ \boldsymbol{x}'_j \end{bmatrix} [\, y_j \quad \boldsymbol{x}_j \,].$$

The algorithm extracted from Fuller (1987) is as follows.

(i) Find the smallest root λ_n of

$$|\boldsymbol{V} - \lambda \boldsymbol{E}| = 0.$$

(ii) Calculate

$$h = \begin{cases} \lambda_n - 2/N & \lambda_n \leq 1 + 1/N \\ 1 - 1/N & \lambda_n > 1 + 1/N. \end{cases}$$

(iii) The unweighted errors-in-variables (EIV) estimator of β is

$$\tilde{\boldsymbol{\beta}} = (\boldsymbol{M}_{XX} - h\boldsymbol{U})^{-1} \boldsymbol{M}_{XY}, \tag{4}$$

where $\boldsymbol{M}_{XX}$ and $\boldsymbol{M}_{XY}$ are defined as in (3), but with $\boldsymbol{X}$ and Y replacing $\boldsymbol{x}$ and y. Equation (4) should be compared to the OLS expression (2).

(iv) The variance of the intrinsic scatter (i.e. the 'noise' which *cannot* be ascribed to measurement error) is estimated by

$$\begin{aligned} \tilde{\sigma_q^2} &= \frac{1}{N-K} \sum_{j=1}^{N} (Y_j - \boldsymbol{X}_j \tilde{\boldsymbol{\beta}})^2 - \frac{1}{N} \sum_{j=1}^{N} \left(\sigma^2_{vj} + \sum_{k=1}^{K} \tilde{\beta}_k^2 \sigma^2_{u,kj} \right) \\ &= \sigma_{\mathrm{e}}^2 - \frac{1}{N} \sum_{j=1}^{N} \left(\sigma^2_{vj} + \sum_{k=1}^{K} \tilde{\beta}_k^2 \sigma^2_{u,kj} \right), \end{aligned} \tag{5}$$

where σ_{e}^2 is the OLS estimate of the intrinsic scatter in the regression: the last factor is evidently a correction for the presence of measurement errors. If the estimate $\tilde{\sigma_q^2}$ is negative, it means that the scatter in the regression can be accounted for entirely in terms of measurement errors, and $\tilde{\sigma_q^2} = 0$.

(v) The total uncertainty in observation j is

$$\sigma_j^2 = \tilde{\sigma_q^2} + \sigma^2_{vj} + \sum_{k=1}^{K} \tilde{\beta}_k \sigma^2_{u,kj},$$

which includes contributions from measurement errors in both dependent and independent variables, as well as the intrinsic scatter q in the regression (1).

(vi) The weighted errors-in-variables (WEIV) estimate of β is

$$\widehat{\boldsymbol{\beta}} = \boldsymbol{H}_{XX}^{-1} \boldsymbol{H}_{XY} \tag{6}$$

where

$$\boldsymbol{H}_{XX} = \sum_{j=1}^{N} \sigma_j^{-2} (\boldsymbol{X}'_j \boldsymbol{X}_j - \Sigma_{uj}) \qquad \boldsymbol{H}_{XY} = \sum_{j=1}^{N} \sigma_j^{-2} \boldsymbol{X}'_j Y_j \ . \tag{7}$$

Equations (6) and (7) can be compared to (4) and (3).

(vii) The large sample covariance matrices of the EIV and WEIV estimators are

$$\boldsymbol{V}(\tilde{\boldsymbol{\beta}}) = (\boldsymbol{M}_{XX} - \boldsymbol{U})^{-1} \left[\sum_{j=1}^{N} \sigma_j^2 \boldsymbol{X}'_j \boldsymbol{X}_j + \sum_{k=1}^{K} \tilde{\beta}_k^2 \sigma_{u,kj}^2 \right] (\boldsymbol{M}_{XX} - \boldsymbol{U})^{-1}$$

$$\boldsymbol{V}(\widehat{\boldsymbol{\beta}}) = \boldsymbol{H}_{XX}^{-1} \left[\sum_{j=1}^{N} \sigma_j^{-2} \boldsymbol{X}'_j \boldsymbol{X}_j + \sum_{k=1}^{K} \sigma_j^{-4} \tilde{\beta}_k^2 \sigma_{u,kj}^2 \right] \boldsymbol{H}_{XX}^{-1}. \tag{8}$$

The standard expression for the covariance matrix of the OLS estimator is

$$\boldsymbol{V}(\boldsymbol{\beta}_{\rm LS}) = \sigma_{\rm e}^2 \boldsymbol{M}_{XX}^{-1}, \tag{9}$$

where the OLS residual variance is defined in (5).

The sample sizes available are relatively small ($34 \leq N \leq 64$; see below), which makes the use of the large-sample covariance matrices (8) for the determination of standard errors suspect. Use is therefore made of a relatively simple non-parametric technique, namely the jackknife (e.g. Neter, Wasserman & Whitmore 1978); this works well for small samples. The method consists of repeating the estimation N times, each time leaving out one of the data points. Denote the result obtained with $(\boldsymbol{X}_j, Y_j)$ omitted by $\boldsymbol{\beta}_{(j)}$, and let

$$\boldsymbol{\beta}_* = \frac{1}{N} \sum_{j=1}^{N} \boldsymbol{\beta}_{(j)} ,$$

the mean of the jackknife estimates. Then, the jackknife estimate of the covariance matrix of $\boldsymbol{\beta}$ is

$$\boldsymbol{V}(\boldsymbol{\beta}) = \frac{N-1}{N} \sum_{j=1}^{N} (\boldsymbol{\beta}_{(j)} - \boldsymbol{\beta}_*)(\boldsymbol{\beta}_{(j)} - \boldsymbol{\beta}_*)'. \tag{10}$$

Standard errors of the components of $\boldsymbol{\beta}$ are, as usual, given by the square roots of the diagonal elements of $\boldsymbol{V}(\boldsymbol{\beta})$.

3 RESULTS

The luminosity is given by

$$L = 4\pi d_{\rm L}^2 P_{\rm bol},$$

where $P_{\rm bol}$ is the bolometric peak flux, and the luminosity distance is

$$d_{\rm L} = \frac{c}{H_0}(1+z) \int_0^z \frac{{\rm d}u}{\sqrt{(1+u)^3 \Omega_M + \Omega_\Lambda}} \equiv \frac{c}{H_0} f(z). \tag{11}$$

For $H_0 = 75\,{\rm km\,s^{-1}\,Mpc^{-1}}$, $c/H_0 = 1.3 \times 10^{28}$ cm and

$$y = \log L = 57.3 + \log P_{\rm bol} + 2 \log f(z) .$$

All data were taken from table 4 of Schaefer (2007). The errors on the logarithmically transformed values were estimated by the delta method:

$$\text{error}[\log_{10} X] \approx \frac{0.4343}{X} \text{error}(X) . \tag{12}$$

In the case of $E_{\rm p}$, some of the errors are asymmetrical; for these the larger of the two errors was used. The redshift terms are not weighted (i.e. these are assumed error free).

All 15 possible combinations of the four luminosity indicators were considered. Table 3 gives the results for all models for which *all* coefficients were significant (i.e. absolute values at least two standard errors) in OLS fitting.

The following can be seen from the table.

(i) All four of the single-variable models were acceptable; so were three of the six two-variable models; and one of the four three-variable models. The single four-variable model was not satisfactory. It is notable that all the multi-variable models have $E_{\rm p}$ as one of the independent variables.

(ii) The intrinsic scatter was found to be zero, i.e. all the residual error could be accounted for by the measurement errors, in models containing $E_{\rm p}$. This is a consequence of the very large measurement errors associated with some observed $E_{\rm p}$: GRB 060108 with $E_{\rm p} = 65^{+600}_{-10}$ is the extreme case. If the six GRB for which error $[\log(E_{\rm p})] > 0.85$ (see 12) are excluded, then non-zero σ_q is obtained.

(iii) There is excellent agreement between the conventional OLS standard errors derived from (9), and the corresponding jackknife estimates from (10). This inspires confidence in the jackknife error estimates for $\tilde{\boldsymbol{\beta}}$ and $\widehat{\boldsymbol{\beta}}$.

(iv) The jackknife procedure involves repeating the model fitting N times, each time omitting one of the data points. Examination of the N individual results then allows one to gauge the effect of leaving out the particular data point. This is a powerful tool in discovering discrepant or influential data elements. The analyses leading to Table 3 did not reveal any data which were excessively influential.

(v) As may have been expected, standard errors for the EIV and WEIV estimates are larger than those associated with the OLS estimates. The latter sets of error estimates are also more realistic, since measurement errors have been taken into account.

Table 3. The results of fitting the OLS model (2) (with Y, $\boldsymbol{X}$ replacing y and $\boldsymbol{x}$), the EIV model (4) and the WEIV model (6) to the data in table 4 of Schaefer (2007). Estimated standard errors are given in brackets: in the case of the OLS estimator, the first value is derived from the conventional covariance matrix (9), while the second is obtained by jackknifing. Standard errors for $\tilde{\boldsymbol{\beta}}$ and $\widehat{\boldsymbol{\beta}}$ are from jackknifing. The last line of information for each model gives the number of GRB with sufficient data for the model to be fitted; the OLS residual scatter; the factor λ_n determining the weighting of the error matrix $\boldsymbol{U}$ in the estimator; and the estimated intrinsic scatter in the regression relation.

Variable	Model		
	$\boldsymbol{\beta}_{\rm LS}$	$\tilde{\boldsymbol{\beta}}$	$\widehat{\boldsymbol{\beta}}$
β_0	51.02 (0.18, 0.19)	50.98 (0.19)	50.96 (0.19)
$(1+z)$	1.89 (0.37, 0.40)	1.86 (0.44)	1.94 (0.43)
$\tau_{\rm L}$	−0.73 (0.09, 0.10)	−0.83 (0.13)	−0.82 (0.10)
	$N = 38$, $\sigma_{\rm e} = 0.37$, $\lambda_n = 2.64$, $\sigma_q = 0.32$		
β_0	53.86 (0.48, 0.46)	57.05 (2.25)	53.62 (0.67)
$(1+z)$	1.84 (0.38, 0.40)	2.21 (0.76)	1.90 (0.76)
V	1.04 (0.21, 0.22)	2.57 (1.10)	0.88 (0.33)
	$N = 51$, $\sigma_{\rm e} = 0.50$, $\lambda_n = 1.10$, $\sigma_q = 0.28$		
β_0	48.64 (0.28, 0.40)	47.67 (0.38)	48.95 (0.68)
$(1+z)$	1.84 (0.27, 0.28)	1.80 (0.32)	2.36 (0.47)
$E_{\rm p}$	1.36 (0.12, 0.17)	1.85 (0.18)	1.25 (0.29)
	$N = 64$, $\sigma_{\rm e} = 0.42$, $\lambda_n = 0.155$, $\sigma_q = 0$		
β_0	51.28 (0.16, 0.16)	51.26 (0.16)	51.29 (0.16)
$(1+z)$	1.62 (0.32, 0.29)	1.62 (0.29)	1.60 (0.29)
$\tau_{\rm R}$	−0.78 (0.11, 0.10)	−0.83 (0.10)	−0.83 (0.11)
	$N = 62$, $\sigma_{\rm e} = 0.46$, $\lambda_n = 7.3$, $\sigma_q = 0.44$		
β_0	49.29 (0.33, 0.39)	48.23 (0.31)	49.17 (0.58)
$(1+z)$	1.98 (0.27, 0.28)	2.04 (0.30)	2.47 (0.43)
$\tau_{\rm L}$	−0.56 (0.07, 0.07)	−0.47 (0.10)	−0.62 (0.14)
$E_{\rm p}$	0.87 (0.15, 0.17)	1.39 (0.17)	0.85 (0.32)
	$N = 36$, $\sigma_{\rm e} = 0.27$, $\lambda_n = 0.345$, $\sigma_q = 0$		
β_0	50.67 (0.57, 0.58)	49.45 (0.83)	50.19 (0.88)
$(1+z)$	2.11 (0.28, 0.29)	2.23 (0.34)	2.77 (0.52)
V	0.70 (0.17, 0.16)	0.60 (0.18)	0.59 (0.38)
$E_{\rm p}$	1.09 (0.15, 0.17)	1.53 (0.31)	1.16 (0.30)
	$N = 48$, $\sigma_{\rm e} = 0.35$, $\lambda_n = 0.115$, $\sigma_q = 0$		
β_0	49.27 (0.26, 0.30)	48.42 (0.40)	49.23 (0.38)
$(1+z)$	1.78 (0.21, 0.21)	1.84 (0.27)	1.99 (0.33)
$E_{\rm p}$	0.98 (0.11, 0.13)	1.39 (0.18)	0.99 (0.18)
$\tau_{\rm R}$	−0.57 (0.08, 0.07)	−0.50 (0.11)	−0.63 (0.14)
	$N = 57$, $\sigma_{\rm e} = 0.31$, $\lambda_n = 0.125$, $\sigma_q = 0$		
β_0	49.31 (0.32, 0.36)	48.40 (0.34)	49.27 (0.45)
$(1+z)$	1.92 (0.25, 0.24)	1.93 (0.27)	2.11 (0.37)
$\tau_{\rm L}$	−0.40 (0.10, 0.07)	−0.27 (0.12)	−0.46 (0.12)
$E_{\rm p}$	0.86 (0.15, 0.17)	1.33 (0.20)	0.82 (0.22)
$\tau_{\rm R}$	−0.29 (0.12, 0.11)	−0.37 (0.14)	−0.37 (0.17)
	$N = 34$, $\sigma_{\rm e} = 0.25$, $\lambda_n = 0.275$, $\sigma_q = 0$		

(vi) The EIV and WEIV estimates are generally not very different from those obtained by OLS. Exceptions are the coefficients of $E_{\rm p}$ obtained by EIV regression, which differ notably from those found by the other two methods, and the EIV coefficient of V in the second model of the table. In any definite calibration, a careful evaluation of the influence of individual data points is evidently called for. The amount of data currently available is not sufficient to warrant this level of detail.

(vii) In theory, the WEIV results are superior to those from EIV, since higher weights are assigned to more accurately determined data. There is, however, an implicit assumption which is hardly ever admitted in weighted regression, namely that the weights [σ_j^{-2} in (7), in this case] are known exactly. In practice, this is rarely (if ever) the case. The use of estimated weights adds additional uncertainty, and may in fact be counterproductive. The reader should therefore bear in mind that WEIV results are predicated on the accuracy of the estimated standard errors of the observables.

Unfortunately, it is not possible to directly compare the above results to those obtained by other authors, primarily due to the differences in the analysis techniques. It is interesting to contrast some of the approaches with that taken above.

(i) The author is aware of only three previous attempts to regress the luminosity on more than one indicator. First, Liang & Zhang (2005) modelled the isotropic equivalent energy $E_{\rm iso}$ in terms of $E_{\rm p}$ and the break time in the optical afterglow light curve. Secondly, Firmani et al. (2006) modelled the luminosity in terms of $E_{\rm p}$ and $T_{0.45}$, the duration of the brightest 45 per cent of the burst. In both cases, the sets of

variables were different from those considered in this paper. Thirdly, Schaefer (2003a) related L to $E_{\rm p}$, $\tau_{\rm L}$ and V. Two separate regressions, of luminosity on $E_{\rm p}$ and $\tau_{\rm L}$, and luminosity on $E_{\rm p}$ and V, were first calculated; a weighted average of the two regressions was then used as the final estimate for L. The estimated parameters do not necessarily satisfy any optimality criterion such as least squares.

(ii) In most studies, the dependence on redshift was constrained, rather than determined from the data. An exception is the paper by Tsutsui et al. (2008) – the authors derived

$$\log L = 50.88(0.07) + 2.53(0.10)\log(1+z) - 0.28(0.03)\log\tau_{\rm L}, \tag{13}$$

where standard errors are given in brackets. The corresponding OLS result in Table 3 is

$$\log L = 51.02(0.18) + 1.89(0.37)\log(1+z) - 0.73(0.09)\log\tau_{\rm L}. \tag{14}$$

The agreement between (13) and (14) is poor, but not as bad as suggested by the small standard errors in (13): these ignore the fact that Tsutsui et al. (2008) made use of estimated rather than measured redshifts. The standard errors evidently do not account for the uncertainties in the redshifts, derived from the Yonetoku et al. (2004) $L - E_{\rm p}$ relation, and are therefore bound to be underestimates.

(iii) The intrinsic scatter in the regression relations has also been calculated by Schaefer (2007) and by Basilakos & Perivolaropoulos (2008). Values of σ_q in Table 3 are consistently lower than those previously determined. Differences can be ascribed to different estimation methods – equation (5) in the case of this paper; versus choosing σ_q such that χ^2 fit statistics were unity, in Schaefer (2007) and Basilakos & Perivolaropoulos (2008).

(iv) There are no counterparts in the literature on GRB luminosity indicators of the bias-adjusted estimates $\tilde{\boldsymbol{\beta}}$ and $\widehat{\boldsymbol{\beta}}$.

4 CONCLUSIONS

The results of this paper strongly suggest that the different light-curve properties of GRBs contain different information about the luminosity, and that there are potentially important gains in combining different indicators. Similar conclusions were reached by, for example, Liang & Zhang (2005) and Firmani et al. (2006). The last model of Table 3 is particularly interesting, as the coefficients of $\tau_{\rm L}$, $E_{\rm p}$ and $\tau_{\rm R}$ are all highly significant.

Unfortunately, the various models in the table cannot be directly compared, as each is based on a different subset of GRBs. This is due to incompleteness in the data set: not all independent variables are available for all the GRBs. For a given selection of independent variables, this necessitated using data for only those GRB for which *all* the variables had been measured. Some research has been done on model selection for incomplete data (see Ziegler 2006). Alternatively, consideration could be given to imputation of missing data, or to estimation methods which use all the available data, i.e. also partial information for some of the GRB (e.g. Schafer 1997; Little & Rubin 2002).

Some insight into the relative merits of the various models can be gained by basing calculations on a subset of the data which are complete. There are 28 GRB satisfying this requirement. Hypothesis testing cannot easily be used to choose between models, as they may be non-nested, i.e. one model is not necessarily a special case of the other. Furthermore, an array of models is considered, and it is required to select one which is, in some sense, optimal. This can be done by using *information criteria*: these are generally the sum of two factors, one of which reflects the goodness-of-fit of the model to the data, while the other is a penalty term for the number of parameters needed to accomplish the fit. The specific form used here is

$$\mathrm{BIC} = N\log\sigma_{\rm e}^2 + K\log N, \tag{15}$$

the Bayes information criterion. The goodness-of-fit is described by $\sigma_{\rm e}^2$, while the terms proportional to K, the number of fitted parameters, is the 'penalty' (see e.g. Burnham & Anderson 2004). Since small values of both $\sigma_{\rm e}^2$ and K are desirable, the 'best' model is that which minimizes the information criterion. The discussion is restricted to the OLS models.

The three best models, according to BIC, are

$$\begin{aligned}
\log L &= 49.18(0.40) + 2.08(0.28)\log(1+z) - 0.44(0.11)\log\tau_{\rm L} + 0.89(0.18)\log E_{\rm p} \\
&\quad - 0.22(0.13)\log\tau_{\rm R} \quad (\mathrm{BIC} = 23.99) \\
\log L &= 49.19(0.42) + 2.15(0.29)\log(1+z) - 0.57(0.09)\log\tau_{\rm L} + 0.86(0.19)\log E_{\rm p} \quad (\mathrm{BIC} = 24.37) \\
\log L &= 50.98(0.19) + 1.88(0.39)\log(1+z) - 0.76(0.11)\log\tau_{\rm L} \quad (\mathrm{BIC} = 39.61).
\end{aligned} \tag{16}$$

Note that the coefficient of $\tau_{\rm R}$ in the optimal model is not significant; this contrasts with the analogous model in Table 3.

The result suggests that the multi-indicator models are preferable, although the conclusion cannot automatically be extended to the models in Table 3. The three models are compared in Fig. 1, which shows a plot of predicted luminosities against the observed values. Examination shows that the circles [second model in (16)] are generally closer to the diagonal line than the triangles [last model in (16)]. The dots [first model in (16)] are generally closer to the line than the circles.

There are a number of issues which may make attempts at physical interpretation of the regression results premature. First of these is the difficulty, referred to in the preceding paragraph, of selecting the 'best' model. It seems pointless to look for physical explanations of the values of the estimated coefficients before it has been established beyond doubt which parameters need to be included. Furthermore, the coefficient of a given variable may be rather different in different models: see Table 4, in which the ranges of estimates are summarized.

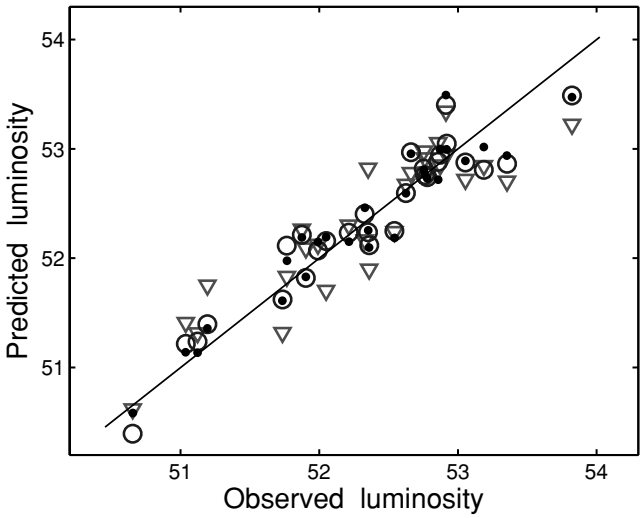

Figure 1. A comparison of the fits of the three models discussed in Section 4. Triangles are for the luminosity indicators (z, $\tau_{\rm L}$); circles for indicators (z, $\tau_{\rm L}$, $E_{\rm p}$) and the dots for the model using (z, $\tau_{\rm L}$, E_P, $\tau_{\rm R}$).

Table 4. The ranges of the WEIV-estimated coefficients across the various models in Table 3. Standard errors are given in brackets.

Parameter	Lowest value	Highest value
$\tau_{\rm L}$	−0.82(0.10)	−0.46(0.12)
V	0.59(0.38)	0.88(0.33)
$E_{\rm p}$	0.82(0.22)	1.25(0.29)
$\tau_{\rm R}$	−0.83(0.11)	−0.37(0.17)

The results in Table 3 demonstrate that differences between $\boldsymbol{\beta}_{\rm LS}$ on the one hand, and $\tilde{\boldsymbol{\beta}}$ or $\widehat{\boldsymbol{\beta}}$ on the other hand, are generally not great compared to the standard errors of the estimated components. This suggests that simple OLS estimates are acceptable, albeit that they may be biased. It should be borne in mind though that the OLS standard errors may be a factor of 2 smaller than the (preferred) EIV or WEIV standard errors. With larger data sets, and hence smaller estimation errors, the differences between the OLS estimates and those obtained from EIV regression will probably be marked enough that use of the latter will be imperative.

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