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Linking multi-epoch CCD photometry of partially overlapping fields

Chris Koen★

Department of Statistics, University of the Western Cape, Private Bag X17, Bellville, 7535 Cape, South Africa

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ABSTRACT

A common problem in CCD photometry is to combine measurements obtained at different epochs. This is especially challenging if the fields only overlap partially, and are sparse, so that not all calibrating stars are observed at all epochs. A least-squares method for determining zero-points for all epochs under these circumstances is formulated. Allowance is made not only for the presence of intra-night measurement errors, but also for epoch-to-epoch scatter in star brightnesses (due, for example, to slow variability or instrumental effects). Expressions are derived for the uncertainties in the estimated zero-points. Three different criteria for selecting optimal calibrating subsets of the available group of stars are introduced. Simulations show that if intra-night variability dominates, it may be best to use all available stars for zero-point determinations. On the other hand, if epoch-to-epoch scatter dominates, smaller subsets of stars may give superior results. Characterization of the uncertainty in the estimated variance of epoch-to-epoch variability remains an important outstanding problem.

Key words: methods: observational – methods: statistical – techniques: photometric.

1 INTRODUCTION

This work arose out of a project designed to study variability of brown dwarfs (e.g. Koen 2013). Target objects were continuously monitored for a few hours in order to test for short time-scale variability. Further runs were obtained at other epochs, either as a follow-up or to also check for longer time-scale brightness changes. The fields of view covered during the different runs often overlapped only partially. Given the typical sparseness of stars in a field, this meant that a very small number of common stars were available to tie photometry from different runs together. In mathematical parlance, the cardinality of the intersection of *all* the sets (from the different nights) of stars is typically quite small.

It seems that a more efficient way of proceeding would be to e.g. use fully the overlapping stars from all *pairs* of starfields. The statistical treatment of this problem is the aim of this paper. Reference will also be made to the 'full-data' case, i.e. starfields which are identical from night to night, as the derived relations simplify to more transparent forms in that special case.

Although the exact problem appears not to have been addressed directly in the literature, there are a number of published papers with varying degrees of relatedness, some of which include observations through multiple filters, the use of standard stars, colour equations, etc. Reed & FitzGerald (1982), Manfroid & Heck (1983), Honeycutt (1992), Padmanabhan et al. (2008) and Regnault et al. (2009), amongst others]. All of these papers make use of least squares for parameter estimation, and some arrive at an explicit matrix equation similar to equation (13) in Section 2 below. There, however, the ways part; the treatment of error estimation in the literature is often unsatisfactory, and the topic of optimally selecting standardizing stars does not seem to have been discussed at all. It should also be noted that the methodology referred to in some of the papers is at times ad hoc – e.g. dealing with missing observations by assigning fictitious values with near-zero weights, rather than formulating the problem to allow variable numbers of observations of different stars.

2 THE STATISTICAL MODEL

The following notation will be used: index the different nights by r ($r = 1, 2, \ldots, R$); the collection of *all* stars observed by s ($s = 1, 2, \ldots, S$); and the individual observations by k ($k = 1, 2, \ldots, n_{rs}$ for a given night r and star s). Measurements are denoted by Y_{rsk}: it is taken that these have been corrected for short time-scale drifts in photometric zero-point, possibly by differential correction. The zero-point for night r is μ_r.

In the ideal situation, exactly the same zero-point applies to all stars; in practice there are small individual offsets for different apparently constant stars. Put differently, the difference in mean magnitudes between any two stars will typically vary by some small amount from night to night. Mechanisms at play have been discussed by Kovács, Bakos & Noyes (2005). The mean offset of star s from μ_r is Δ_{rs}.

The crucial part of the statistical model of this paper is the assumption that, for a given star s, the Δ_{rs} are distributed randomly, with variance $\sigma^2_{\eta s}$, around an overall mean Δ_s:

$$\Delta_{rs} = \Delta_s + \eta_{rs}\ , \qquad \mathrm{var}(\eta_{rs}) = \sigma^2_{\eta s}\ , \tag{1}$$

where the mean of the random numbers η_{rs} (over different nights r) is zero.

★ E-mail: ckoen@uwc.ac.za

The overall model for a measurement of a constant star is then

$$Y_{rsk} = \mu_r + \Delta_{rs} + \epsilon_{rsk}$$
$$= \mu_r + \Delta_s + \eta_{rs} + \epsilon_{rsk} \ , \quad (2)$$

where ϵ_{rsk} is the random variation for an individual measurement of star s on night r, assumed to have zero mean and variance σ^2_{rs}. (It is relatively easy to extend the theory to allow for these variances to depend on the index k, and for the ϵ_{rsk} with different indices s and/or k to be correlated. In order to keep the exposition as clear as possible, it is not done here). The σ^2_{rs} are easily estimated from the raw observations:

$$\widehat{\sigma}^2_{rs} = \frac{1}{(n_{rs}-1)} \sum_{k=1}^{n_{rs}} (Y_{rsk} - Y_{rs\bullet})^2, \quad (3)$$

where $Y_{rs\bullet}$ is the mean of Y_{rsk} over all observations k. Model (2) can then be replaced by

$$Y_{rs\bullet} = \mu_r + \Delta_s + e_{rs}, \quad (4)$$

where

$$e_{rs} \equiv \eta_{rs} + \frac{1}{n_{rs}} \sum_k \epsilon_{rsk} \ . \quad (5)$$

It is noted in passing that equation (5) constitutes one of the novelties in this paper: since the aim is to combine sets of time series photometry obtained at different epochs, there is additional information available in the form of precise estimates, in the form of (3), of the *measurement errors* at each epoch. Furthermore, outlying individual measurements can be removed to reduce the variance in (3).

The immediate aim is to tie photometry from the different nights together, i.e. to estimate μ_r. Since we are not attempting to place these on an absolute footing, an arbitrary constant can be added to each of the μ_i without changing the essential results. One way of dealing with this indeterminacy is to fix one of the μ_r; the notation is simplest if

$$\mu_R = 0$$

is selected (see also e.g. Reed & FitzGerald 1982, Honeycutt 1992).

A least-squares estimating equation follows from (4): minimize

$$SS = \sum_{r=1}^{R} \sum_s (Y_{rs\bullet} - \mu_r - \Delta_s)^2$$
$$= \sum_{r=1}^{R-1} \sum_s (Y_{rs\bullet} - \mu_r - \Delta_s)^2 + \sum_s (Y_{Rs\bullet} - \Delta_s)^2 \quad (6)$$

with respect to μ_r and Δ_s. Note that if the summation over k had been retained, i.e. if (6) had been written in terms of Y_{rsk} rather than $Y_{rs\bullet}$, terms with the same indices r and s, but different k, would have been correlated because of sharing the same η_{rs}. This would have required some adjustment (multiplication of a vector of terms by the inverse square root of its covariance matrix).

Since not all stars are necessarily observed every night, it is understood that summation in (6) over s involves only values appropriate for a given r. The notation $\mathcal{R}_s$ will be used for the set of nights during which star s was observed. The set of stars observed during night r will similarly be denoted by $\mathcal{S}_r$. Later calculations will require that the last night sometimes be excluded from $\mathcal{R}_s$ – in such cases $\mathcal{R}'_s = \mathcal{R}_s \setminus R$ (indices $\mathcal{R}_s$ excluding index R).

A point worth remarking on is that many previous papers (see references in Section 1) make use of *weighted* least squares. In theory, weighting observations produces more accurate parameter estimates. In practice, the weights used are usually themselves *estimates*, with accompanying uncertainty. For data sets as small as those considered in this paper, this added uncertainty is counterproductive.

Taking partial derivatives of (6) and setting the results equal to zero,

$$\sum_{s \in \mathcal{S}_r} (Y_{rs\bullet} - \mu_r - \Delta_s) = 0 \qquad r = 1, 2, \ldots, R-1$$
$$\sum_{r \in \mathcal{R}_s} (Y_{rs\bullet} - \mu_r - \Delta_s) = 0 \qquad s = 1, 2, \ldots, S \quad (7)$$

follows, where it is understood that $\mu_R = 0$ in the second equation. Let

$$U_r = \sum_{s \in \mathcal{S}_r} Y_{rs\bullet} \qquad V_s = \sum_{r \in \mathcal{R}_s} Y_{rs\bullet}$$
$$N_r = \#\{\mathcal{S}_r\} \qquad M_s = \#\{\mathcal{R}_s\}. \quad (8)$$

Here N_r (the cardinality of the set $\mathcal{S}_r$) is the number of stars observed during night r, while M_s is the number of nights during which star s was observed.

With the help of (8), equations (7) can be written in matrix form as

$$\mathbf{D}_2 \boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{\Delta} = \boldsymbol{U}$$
$$\mathbf{D}_1 \boldsymbol{\Delta} + \boldsymbol{A}'\boldsymbol{\mu} = \boldsymbol{V} \ . \quad (9)$$

In (9), $\mathbf{D}_1$ and $\mathbf{D}_2$ are diagonal matrices of orders S and $R-1$, with, respectively, the M_s and N_r on the diagonals; $\boldsymbol{U}$ and $\boldsymbol{V}$ are column vectors with respective entries U_r and V_s, and the entries in $\boldsymbol{A}$ are defined by

$$A(r, s) = \begin{cases} 0 & n_{rs} = 0 \\ 1 & n_{rs} \neq 0 \end{cases} \quad (10)$$

for $r = 1, 2, \ldots, R-1$ and $s = 1, 2, \ldots, S$. It is also useful to define

$$\boldsymbol{E} = \mathbf{D}_2 - \boldsymbol{A}\mathbf{D}_1^{-1}\boldsymbol{A}'$$
$$\boldsymbol{b} = \boldsymbol{U} - \boldsymbol{A}\mathbf{D}_1^{-1}\boldsymbol{V}. \quad (11)$$

The solution of (9) is then easily shown to be

$$\widehat{\boldsymbol{\mu}} = \boldsymbol{E}^{-1}\boldsymbol{b}$$
$$\widehat{\boldsymbol{\Delta}} = \mathbf{D}_1^{-1}[\boldsymbol{V} - \boldsymbol{A}'\widehat{\boldsymbol{\mu}}]. \quad (12)$$

The estimation problem (9) can also be written as the single matrix equation:

$$\boldsymbol{G} \begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\Delta} \end{bmatrix} = \boldsymbol{W}$$

with

$$\boldsymbol{G} = \begin{bmatrix} \mathbf{D}_2 & \boldsymbol{A} \\ \boldsymbol{A}' & \mathbf{D}_1 \end{bmatrix} \qquad \boldsymbol{W} = \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} \widehat{\boldsymbol{\mu}} \\ \widehat{\boldsymbol{\Delta}} \end{bmatrix} = \boldsymbol{G}^{-1}\boldsymbol{W}. \quad (13)$$

If exactly the same set of stars is observed every night, then $N_r \equiv S$ and $M_s \equiv R$, and $\boldsymbol{A} = \boldsymbol{J}_{R-1,S}$ [an $(R-1) \times S$ matrix with all entries equal to unity]. It can then be shown that

$$\boldsymbol{G}^{-1} = \frac{1}{S}\begin{bmatrix} \boldsymbol{G}_1 & \boldsymbol{G}_2 \\ \boldsymbol{G}_2' & \boldsymbol{G}_3 \end{bmatrix}, \qquad (14)$$

with

$$\boldsymbol{G}_1 = \boldsymbol{I}_{R-1} + \boldsymbol{J}_{R-1,R-1} \qquad \boldsymbol{G}_2 = -\boldsymbol{J}_{R-1,S}$$

$$\boldsymbol{G}_3 = \frac{S}{R}\boldsymbol{I}_S + \frac{R-1}{R}\boldsymbol{J}_{S,S},$$

where $\boldsymbol{I}_S$ is the $S \times S$ identity matrix. Solution (13) can then be written explicitly in terms of the observations:

$$\widehat{\mu}_r = Y_{r\bullet\bullet} - Y_{R\bullet\bullet} = \frac{1}{S}\sum_{s=1}^{S} Y_{rs\bullet} - \frac{1}{S}\sum_{s=1}^{S} Y_{Rs\bullet}$$

$$\widehat{\Delta}_s = Y_{\bullet s\bullet} - Y_{\bullet\bullet\bullet} + Y_{R\bullet\bullet}$$

$$= \frac{1}{R}\sum_{r=1}^{R} Y_{rs\bullet} - \frac{1}{R}\frac{1}{S}\sum_{r=1}^{R}\sum_{s=1}^{S} Y_{rs\bullet} + \frac{1}{S}\sum_{s=1}^{S} Y_{Rs\bullet}. \qquad (15)$$

3 STANDARD ERRORS OF THE ESTIMATES

It follows from (13) that the covariance matrix of $[\widehat{\boldsymbol{\mu}} \;\; \widehat{\boldsymbol{\Delta}}]'$ is

$$\mathbf{C} = \boldsymbol{G}^{-1}\mathrm{cov}(\boldsymbol{W}, \boldsymbol{W}')\boldsymbol{G}^{-1} = \boldsymbol{G}^{-1}\boldsymbol{H}\boldsymbol{G}^{-1}. \qquad (16)$$

The entries in the covariance matrix $\mathbf{H}$ are

$$H_{ij} = \mathrm{cov}(U_i, U_j) = \mathrm{cov}\left(\sum_k e_{ik}, \sum_\ell e_{j\ell}\right)$$
$$= \delta_{ij}\sum_{k\in\mathcal{S}_i}\mathrm{cov}(e_{ik}, e_{ik})$$
$$= \delta_{ij}\sum_{k\in\mathcal{S}_i}\mathrm{var}(e_{ik})$$
$$i, j = 1, 2, \ldots, R-1$$

$$H_{R-1+i,R-1+j} = \mathrm{cov}(V_i, V_j) = \mathrm{cov}\left(\sum_k e_{ki}, \sum_\ell e_{\ell j}\right)$$
$$= \delta_{ij}\sum_{k\in\mathcal{R}_j}\mathrm{cov}(e_{kj}, e_{kj})$$
$$= \delta_{ij}\sum_{k\in\mathcal{R}_j}\mathrm{var}(e_{kj})$$
$$i, j = 1, 2, \ldots, S$$

$$H_{i,R-1+j} = H_{R-1+j,i} = \mathrm{cov}(U_i, V_j) = \mathrm{cov}\left(\sum_k e_{ik}, \sum_\ell e_{\ell j}\right)$$
$$= \sum_{k\in\mathcal{S}_i}\sum_{\ell\in\mathcal{R}_j}\mathrm{cov}(e_{ik}, e_{\ell j})$$
$$= \delta(j \in \mathcal{S}_i)\mathrm{var}(e_{ij})$$
$$= A(i, j)\mathrm{var}(e_{ij})$$
$$i = 1, 2, \ldots, R-1;$$
$$j = 1, 2, \ldots, S. \qquad (17)$$

In these equations δ is the Kronecker delta function. The $A(i, j)$ are defined in (10).

If all stars are observed on every night,

$$\mathrm{cov}(\widehat{\mu}_i, \widehat{\mu}_j) = \frac{1}{S^2}\sum_{\ell=1}^{S}\left[(1+\delta_{ij})\sigma_{\eta\ell}^2 + \frac{\delta_{ij}}{n_i}\sigma_{i\ell}^2 + \frac{1}{n_R}\sigma_{R\ell}^2\right]$$

$$\mathrm{cov}(\widehat{\Delta}_i, \widehat{\Delta}_j) = \frac{\delta_{ij}}{R^2}\left[R\sigma_{\eta i}^2 + \sum_{\ell=1}^{R}\frac{\sigma_{\ell i}^2}{n_\ell}\right] + \frac{\sigma_{Ri}^2 + \sigma_{Rj}^2}{RSn_R}$$
$$+ \frac{R-1}{RS^2}\sum_{\ell=1}^{S}(\sigma_{\eta\ell}^2 + \sigma_{R\ell}^2) - \frac{1}{R^2 S}\sum_{\ell=1}^{R}\frac{\sigma_{\ell i}^2 + \sigma_{\ell j}^2}{n_\ell}$$
$$+ \frac{1}{R^2 S^2}\sum_{k=1}^{R}\sum_{\ell=1}^{S}\frac{\sigma_{k\ell}^2}{n_k}$$

$$\mathrm{cov}(\widehat{\mu}_i, \widehat{\Delta}_j) = \frac{1}{RS}\left(\frac{\sigma_{ij}^2}{n_i} - \frac{\sigma_{Rj}^2}{n_R}\right) - \frac{1}{S^2}\sum_{\ell=1}^{S}\sigma_{\eta\ell}^2$$
$$+ \frac{1-R}{RS^2 n_R}\sum_{\ell=1}^{S}\sigma_{R\ell}^2 - \frac{1}{RS^2 n_i}\sum_{\ell=1}^{S}\sigma_{i\ell}^2, \qquad (18)$$

where n_i is the number of individual measurements obtained during night i. Of course, (18) could also be derived directly from the explicit formulae in (15).

Note that for $i = j$ the first of equations (18) can be written as

$$\mathrm{var}(\widehat{\mu}_i) = \gamma_{i\bullet}/S, \qquad (19)$$

where the mean variance $\gamma_{i\bullet}$ is defined as

$$\gamma_{i\bullet} = \frac{1}{S}\sum_{s=1}^{S}\gamma_{is} = \frac{1}{S}\sum_{s=1}^{S}\left(2\sigma_{\eta s}^2 + \frac{\sigma_{is}^2}{n_i} + \frac{\sigma_{Rs}^2}{n_R}\right). \qquad (20)$$

Standard errors of the μ_r and Δ_s are given by square roots of the diagonal elements of $\mathbf{C}$ in (16), but require variances of the e_{ij} (see equation 17). From (5),

$$\mathrm{var}(e_{rs}) = \mathrm{var}(\eta_s) + \frac{1}{n_{rs}}\sigma_{rs}^2.$$

The last term is easily estimated from the individual time series measurements (cf. equation 3). Estimated variances of the η_s can then be calculated as described in Appendix A [fully observed data sets, i.e. $A(r, s) = 1$ for all r, s] or Appendix B [some $A(r, s) = 0$].

Estimates obtained from (A7) or (B10) have the virtue that they are unbiased. However, unless the number of nights R is large, obtaining $\widehat{\sigma}_{\eta s}^2 < 0$ is a distinct possibility. Similar problems are encountered in genetics research, where the number of genes (S) may be large, but the number of replications (R) is small, and hence accurate estimation of variance components is an issue (e.g. Tong & Wang 2007). A possible cure would be 'shrinkage', in essence using a weighted mean of $\widehat{\sigma}_{\eta s}^2$ and the overall mean variance $\sum_k \widehat{\sigma}_{\eta k}^2/S$. Two simple alternatives are to either replace negative estimates by zeros or to assume that σ_η is the same for all stars. Estimation formulae for the latter case are given in equations (A8) and (B11).

Study of equations (13) and (8) reveals that the estimators $\widehat{\mu}_r$ and $\widehat{\Delta}_s$ are linear combinations of the $Y_{rs\bullet}$. This is particularly obvious in the case of fully observed data – see (15). If the $Y_{rs\bullet}$ are Gaussian, so are the estimators. It then follows that the joint distribution of all the estimates $\widehat{\mu}_r$ and $\widehat{\Delta}_s$ is multivariate Gaussian, with the mean given by (13), and the covariance matrix by (16) and (17).

4 WHICH STARS SHOULD BE INCLUDED?

It is, of course, desirable to have the variances of the μ_i as small as possible. This suggests (i) judiciously choosing the set of stars included, and (ii) giving thought as to which nightly μ_r to assign the zero value, i.e. the choice of r to be set equal to R. An answer to point (ii) is suggested by equation (20): refer all measurements to the night r for which

$$\sum_s \widehat{\sigma}_{rs}^2/n_r$$

is a minimum.

Equation (20) applies to the case of full data; if not all stars are measured during all nights r, the situation could be more complicated. In particular, in the full-data case, *all* the variances of all the μ_r would be reduced by a wise choice of the index R, but in the case of partially overlapping starfields minimization of

$$\sum_{s \in \mathcal{S}_r} \widehat{\sigma}_{rs}^2/n_{rs}$$

may be inappropriate, as different $\mathcal{S}_r$ will contain different stars.

In the statistics literature, a number of approaches to this type of experimental design problem have been proposed. Two standard criteria which are appropriate to the present context are 'A-optimality'

$$\text{Minimize} \sum_{r=1}^{R-1} \text{var}(\widehat{\mu}_r) = \text{Minimize} \sum_{r=1}^{R-1} \boldsymbol{C}_{rr} \tag{21}$$

and 'D-optimality'

$$\text{Minimize } |\boldsymbol{C}_{R-1}|$$

or, more conveniently,

$$\text{Minimize } \log |\boldsymbol{C}_{R-1}|, \tag{22}$$

where $\boldsymbol{C}_{R-1}$ is the upper-left $(R-1) \times (R-1)$ part of the covariance matrix $\mathbf{C}$ in (16). Confidence volumes (the multi-dimensional generalization of confidence intervals) for the collection of μ_r are proportional to the determinant in (22); hence, minimization of $|\boldsymbol{C}_{R-1}|$ is equivalent to ensuring accurate determination of the full set of μ_r (see e.g. Myers & Montgomery 1995). A third possibility is

$$\text{Minimize} \left\{ \max_r [\text{var}(\widehat{\mu}_r)] \right\} , \tag{23}$$

which is related to 'E-optimality' in the literature.

To summarize: different subsets of calibrating stars are tried, and the optimality criterion of choice is calculated for each subset. Furthermore, for a given subset of stars, each night is in turn considered as reference point. 'Optimality' in the present context clearly involves minimizing the zero-point uncertainties in some way, but this could take different forms.

(i) Minimize the *mean* uncertainty across all estimated μ_r ($r = 1, 2, \ldots, R$). This is conceptually simple but ignores the fact that the $\widehat{\mu}_r$ may be correlated.

(ii) Explicitly take into account the inter-relationship between the $\widehat{\mu}_r$, and minimize the multi-dimensional confidence region for the entire collection of estimated zero-points. This is mathematically more involved, and the results do not have as simple an interpretation as in (i).

(iii) Choose the calibrating stars such that the largest possible variance of any $\widehat{\mu}_r$ is as small as possible. In this case, the mean variance may be larger than that in (i), and/or the confidence volume greater than that in (ii), but the worst case $\widehat{\mu}_r$ will be a minimum.

In general, there is nothing which obviously commends one of the criteria over the other two, but it may be that the context favours a particular form. The different approaches are probably best seen as alternative sensible procedures to get 'good' sets of calibrating stars.

Some insight into the operation of criteria (21) and (23) can be gained by considering the full-data case with *known* variances γ_{ij} and proceeding from the first of equations (18). [In practice, this would imply the data from many nights being available, i.e. $R \gg 1$.] Consider night i, such that $\text{var}(\widehat{\mu}_i) \geq \text{var}(\widehat{\mu}_j)$, for $j = 1, 2, \ldots, R$ (i.e. night i has the largest zero-point uncertainty). It is not difficult to show that $\text{var}(\widehat{\mu}_i)$ will be reduced by exclusion of star k if

$$\gamma_{ik} > \left(2 + \frac{1}{S-1}\right) \frac{1}{S-1} \sum_{s \neq k}^{S} \gamma_{is} = \left(2 + \frac{1}{S-1}\right) \overline{\gamma}_{i(k)}, \tag{24}$$

where the notation $\overline{\gamma}_{i(k)}$ indicates an average taken over all stars excluding star k, and the variance γ_{ik} is as defined in (20). In other words, star k is a liability as far as criterion (23) is concerned if the variance γ_{ik} associated with it is more than about twice the average of the other stars' variances. The condition required by criterion (21) is very similar:

$$\gamma_{\bullet k} > \left(2 + \frac{1}{S-1}\right) \overline{\gamma}_{\bullet(k)}, \tag{25}$$

i.e. the average of (24) over all nights i. The implication is that stars with outlying large γ_{ij} do not contribute accuracy in the estimation of the μ_i. Note the further implication that if there are a few stars, with γ_{ij} widely different for different j, then criteria (21) and (23) are likely to select a small number of stars with the smallest variances.

In practice, (24) and (25) are overly stringent, since allowance was not made for the possibility of there being *different* optimal indices R for the two sets of stars $\{1, 2, \ldots, k-1, k, k+1, \ldots, S\}$ and $\{1, 2, \ldots, k-1, k+1, \ldots, S-1\}$.

If $n_{ij} \gg 1$, and/or the scatter σ_{ij} in the light curves is small (e.g. if the standardizing stars are bright), then the covariances of the $\widehat{\mu}_r$ will be dominated by the $\sigma_{\eta s}$, and hence will not depend on the night r. In that case, the three criteria (21)–(23) will select the same set of stars, at least for fully observed data.

5 ILLUSTRATIVE SIMULATION RESULTS

Fig. 1 illustrates the material presented at the end of Section 4. It is mostly based on the following configuration: the $\sigma_{\eta s}$ are uniformly distributed over (0.05, 0.5) mag; σ_{ij} are uniform over (0.02, 0.1) mag; and it is assumed that a single observation of each star is obtained during each night ($n_{ij} = 1$, for all i, j). For the simulations summarized by the dotted line the upper limit on $\sigma_{\eta s}$ was increased from 0.5 to 0.8 mag. The number of nights is $R = 20$. Note that for the purpose of selecting the best subsample of stars, it is not necessary to estimate the μ_r and Δ_s – knowing the upper-left $(R - 1) \times (R - 1)$ part of the covariance $\mathbf{C}$ is sufficient. For now it is assumed that the $\sigma_{\eta s}$ and σ_{ij} needed to calculate $\mathbf{C}$ are known.

The broken lines in Fig. 1 trace the decrease in the maximal variance in $\widehat{\mu}_r$, over all nights r, as stars contributing most to the variance are removed one at a time. The lines represent averages, taken over 200 independent simulations. For convenience, plotted variances have been multiplied by 1000. Starting from a sample of size 50, the average minimum is at about 15: further reducing S leads to a steadily increasing variance. For a smaller initial sample size of 25, the minimum is, on average, at about $S = 8$.

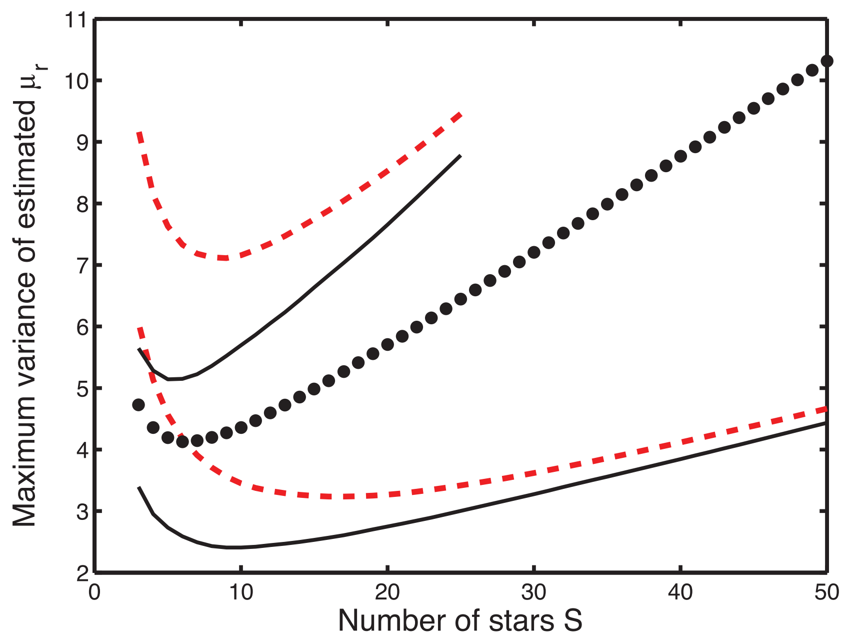

Figure 1. Simulated results illustrating the selection of stars to include in the sample used to estimate the nightly offsets μ_r, in the case where the $\sigma_{\eta s}$ are the main contributors to the variances of the $\widehat{\mu}_r$. The broken lines show the decrease in the criterion (23) by successively removing the star with the greatest contribution to the maximum variance over all μ_r, starting from full samples of sizes 50 and 25, respectively. The solid lines show the additional benefit of re-determining the optimal reference night at each step. The dotted line gives the results of increasing the upper bound on the $\sigma_{\eta s}$; see the text for more details. For convenience, the plotted variances have been scaled by a factor of 1000.

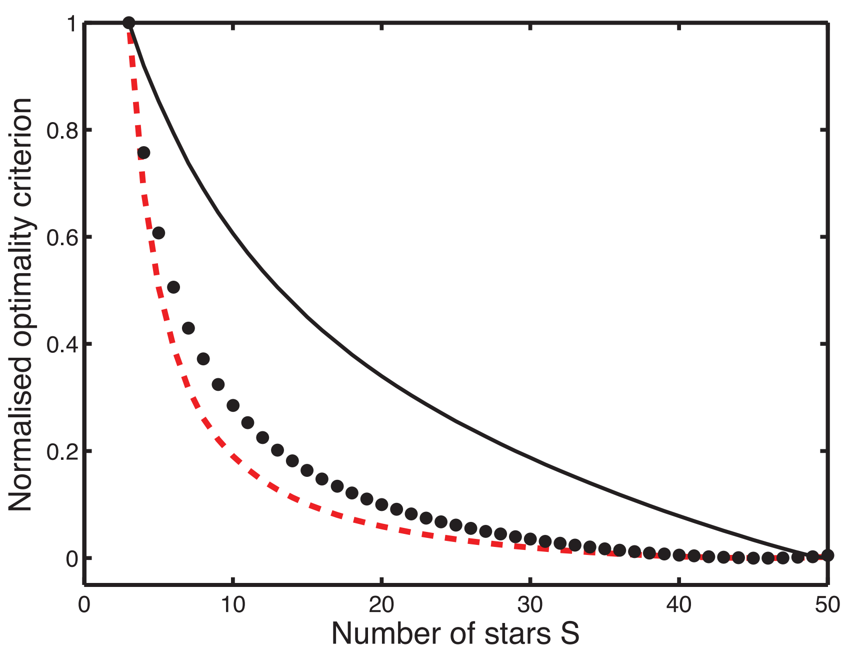

Figure 2. Simulated results illustrating the selection of stars to include in the sample used to estimate the nightly offsets μ_r, in the case where the σ_{ij} are the main contributors to the variances of the $\widehat{\mu}_r$. The dotted, solid and broken lines, respectively, show the normalized results for criteria (21), (22) and (23), as functions of the number of stars remaining in the sample.

The substantial advantage of choosing anew, at each step, the optimal reference night $r = R$ is demonstrated by the two solid lines. It is interesting that the minima of the two solid curves are also at smaller S than their broken-line counterparts. The dotted line, with higher average $\sigma_{\eta s}$, may be compared with the lower solid line, since the optimal reference night was also determined afresh each time a star was removed from the sample. It is of no surprise that the decrease in the dotted line with decreasing S is steeper; note also that the minimum is reached at a slightly lower value of S than in the case with $\max(\sigma_{\eta s}) = 0.5$ mag.

Since the covariances of the $\widehat{\mu}_r$ are largely determined by the $\sigma_{\eta s}$, with the σ_{ij} playing only a minor role, very similar results are obtained if either of the criteria (21) or (22) is used.

Exchanging the distributions of the $\sigma_{\eta s}$ and σ_{rs}, i.e. having the within-night scatter dominate, leads to the results plotted in Fig. 2 (again showing mean values of 200 simulations in each case). Relative changes in criteria (21) and (23) are small when the full sample is initially reduced, but increase sharply as sample sizes drop below 10. In all three cases, the reference index R was adjusted at each step to further minimize the relevant criterion.

Since the primary aim of Fig. 2 is to compare the shapes of the three curves, these were normalized. Actual ranges may also be of interest, at least for criteria (23) [$(1.0, 12.8) \times 10^{-3}$] and (21) [$(17.0, 109.4) \times 10^{-3}$]. Clearly, if the intra-night errors are dominant, it could be desirable to include as many stars in the sample as possible.

The difference in the functional dependence on S in Figs 1 and 2 can be understood in terms of equation (25), and the fact that for dominant σ_{ij} there are R times as many potentially large variances as when the $\sigma_{\eta s}$ are dominant.

In real applications, the distribution of σ_{ij} is, of course, unlikely to be uniform. In particular, there is often a small number of brighter stars for which measurements are exceptionally accurate. Given the form

$$\gamma_{is} = 2\sigma_{\eta s}^2 + \frac{\sigma_{is}^2}{n_i} + \frac{\sigma_{Rs}^2}{n_R}$$

of the relevant combined variance [see (18)–(20)], this will only be important if $\sigma_{\eta s}$ also happens to be small for those stars. Simulations show in such cases a lowered rate at which optimality criteria increase with decreasing subset size. This can be explained by the increased prominence of the high-accuracy measurements as the set of calibrating stars is made smaller.

Next we turn to simulations in which there are missing values, i.e. not all stars are observed every night. In order to avoid completely unrealistic data sets, the restrictions $M_s \geq 2, N_r \geq 1$ are imposed. (In practice, $N_r = 1$ is not very realistic, but the assumption of known $\sigma_{\eta s}$ and σ_{ij} mitigates this case. The practical situation in which these variances are estimated from the data, rather than specified, is addressed below.) The results of simulations based on the same configuration as that leading to the bottom curve in Fig. 1, but with 30 per cent missing observations, are plotted in Fig. 3 (top panel). The outcomes are slightly different if criteria (21) and (22) are used; hence, these are also shown, in the other two panels of the figure. The general shapes are similar to those in Fig. 1, although the minima are somewhat wider.

Data analogous to those giving rise to Fig. 2, but with 30 per cent of the data randomly missing, were also simulated. The results were quite similar in appearance to those in Fig. 2, and are hence not shown.

As was pointed out at the end of Section 3, the distribution of the $\widehat{\mu}_r$ and $\widehat{\Delta}_s$ is fully known, at least in terms of the σ_{ij} and $\sigma_{\eta s}$. The distribution of the estimated $\sigma_{\eta s}^2$ (Appendices A and B) is a different kettle of fish: it is a linear combination of squared Gaussian random variates, which are interdependent and have different variances. Although a formula for the variance of $\widehat{\sigma}_{\eta s}^2$ can be worked out, it requires considerable algebra. Here, we restrict ourselves to reporting the results of a number of computer experiments.

(i) If σ_{ij} is comparable to, or larger than, $\sigma_{\eta s}$, then estimates of the latter are systematically too small, and may be very inaccurate. This is not necessarily a problem in the context of this paper, as the principal use of these variances is in the determination of the covariance matrix **C** through the matrix **H** in (17): if σ_{ij} is dominant, then poorly determined $\sigma_{\eta s}$ will have a relatively minor effect on the accuracy of **C**. In what follows, it is assumed that random variability is dominated by η_{rs} in (1), i.e. that $\sigma_{\eta s} \gg \sigma_{rs}/n_{rs}$ for all stars s and

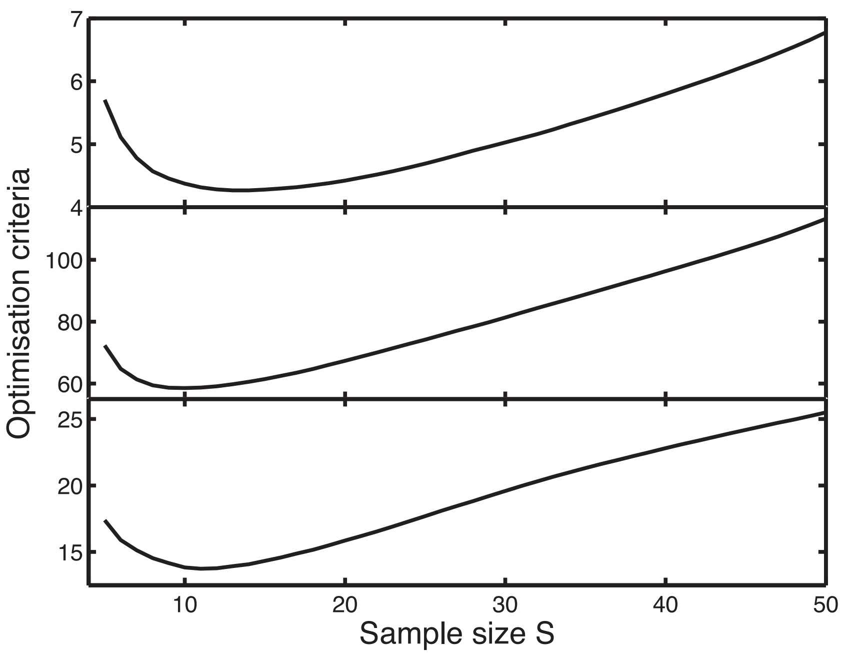

Figure 3. As for the bottom solid line in Fig. 1, but with two differences: (i) 30 per cent of the observations are missing (randomly across nights and stars), and (ii) results for all three criteria are shown. From top to bottom, equations (23), (21) and (22). The covariance matrix **C** has been multiplied by 1000, and the determinant has been log-transformed.

nights r. The simulations reported below were obtained with $\sigma_{rs} = 0$.

(ii) Fig. 4 shows histograms of $\widehat{\sigma}^2_\eta$, for true values $\sigma^2_\eta =$ 0.0025, 0.029, 0.096 and 0.25. The 10 000 simulations were based on $R = S = 20$. Quantile–quantile comparison plots confirm the visual impression that the densities are generally similar in shape, except possibly in the extreme tails. The same does not hold true for smaller R, for which the distributions corresponding to larger σ^2_η are more asymmetric, with short lower tails.

(iii) For a given value of R, the standard deviation of $\widehat{\sigma}^2_\eta$ is proportional to σ^2_η, to a very good approximation. The slope of the linear relation depends on R; the equation

$$\text{std.dev.}\left(\widehat{\sigma}^2_\eta\right) \approx 1.762 R^{-0.555} \sigma^2_\eta$$

is accurate to a few millimagnitudes for complete data sets, provided R and S are larger than 10 or so.

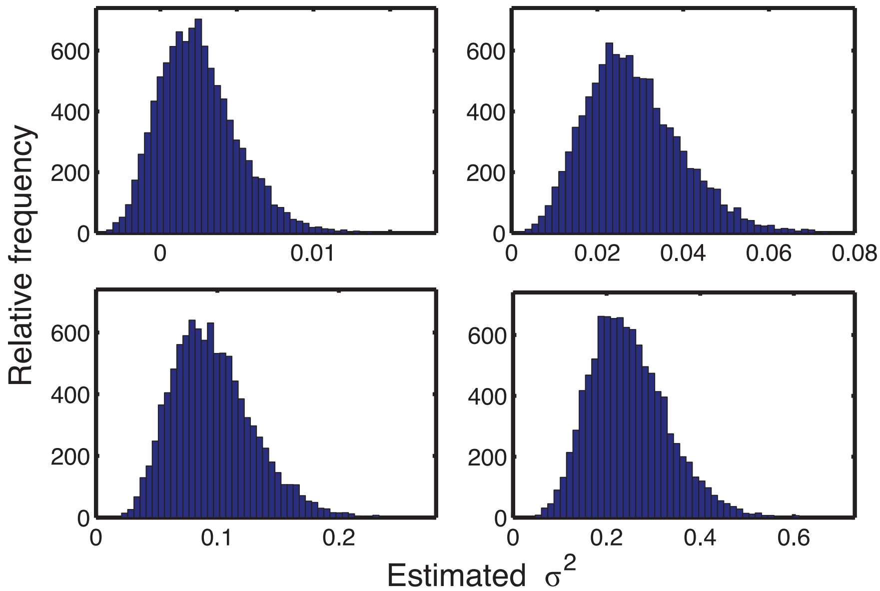

Figure 4. Histograms of $\widehat{\sigma}^2_\eta$, where the true parameter values are (anti-clockwise, starting at the top left) 0.0025, 0.029, 0.25 and 0.096. The figure summarizes the results of 10 000 simulation trials with $R = S = 20$.

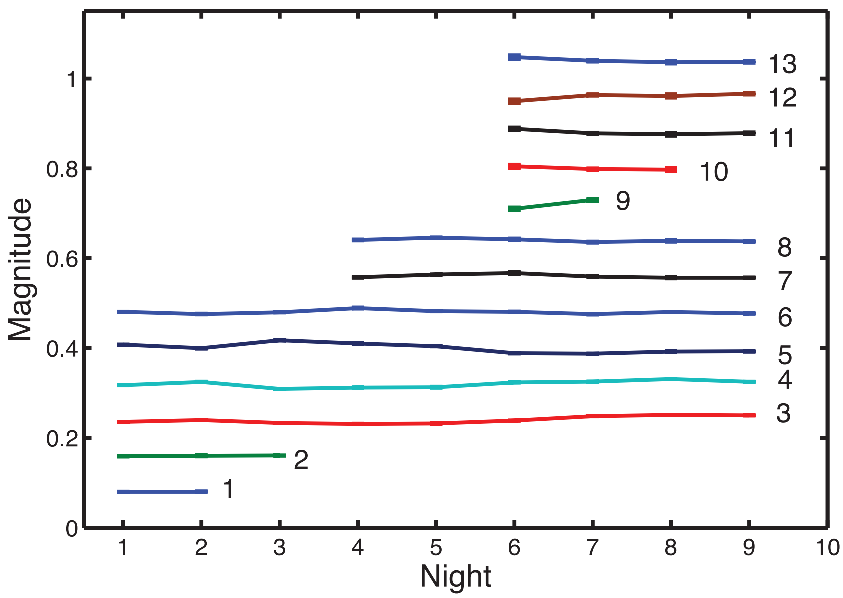

Figure 5. A data set (I-band observations of 2M 1155−3727) which is typical of the type to which the theory in this paper can be applied. Only four stars (3–6) were measured on all nine nights. Each light curve is plotted with an arbitrary zero-point, and an adjustment has also been made for night-to-night zero-point shifts common to all stars.

6 ILLUSTRATIVE RESULTS FOR TWO REAL DATA SETS

Fig. 5 shows the results of *I*-band observations of the ultracool dwarf (UCD) 2MASS J1155395−372735 (hereafter 2M 1155−3727). Thirteen stars in its field of view were measured at least twice, but only four of these were observed during all nine nights. The number of observations obtained during each night varied from 32 to 212, and the standard deviation of the scatter around the nightly light curves was in the range $1.8 \le \sigma_{ij} \le 31.3$ mmag. It follows that the standard errors of the plotted points are in the range $0.23 \le \sigma_{ij}/\sqrt{n_{ij}} \le 4.2$ mmag; these are indicated by error bars, barely visible in the plot. The fraction of 'missing values' is 0.40.

The behaviour of the three criteria (21)–(23) with decreasing S is summarized by Fig. 6. Here, and in the second example below, the index R was also adjusted at each step in order to optimize results.

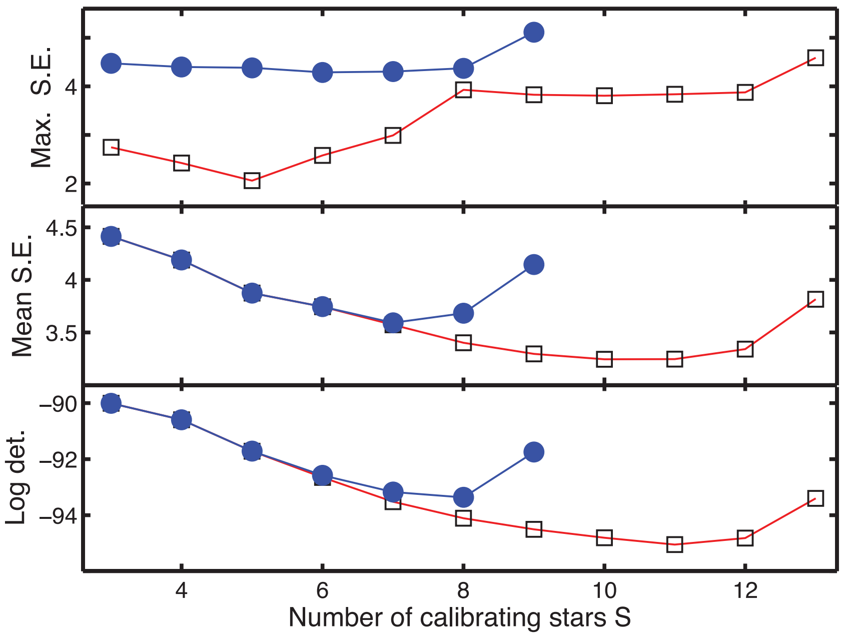

Figure 6. The selection criteria (21)–(23) for the data in Fig. 5, as a function of the number S of stars included in the calibrating subset. The criteria plotted in the top two panels have been scaled so that these are, respectively, the maximum standard error of any μ_r, and the mean standard error over all μ_r (in millimagnitudes). The squares and filled circles, respectively, show results starting from the full complement of 13 stars, and from only the 9 stars measured on at least four occasions ($M_s \ge 4$).

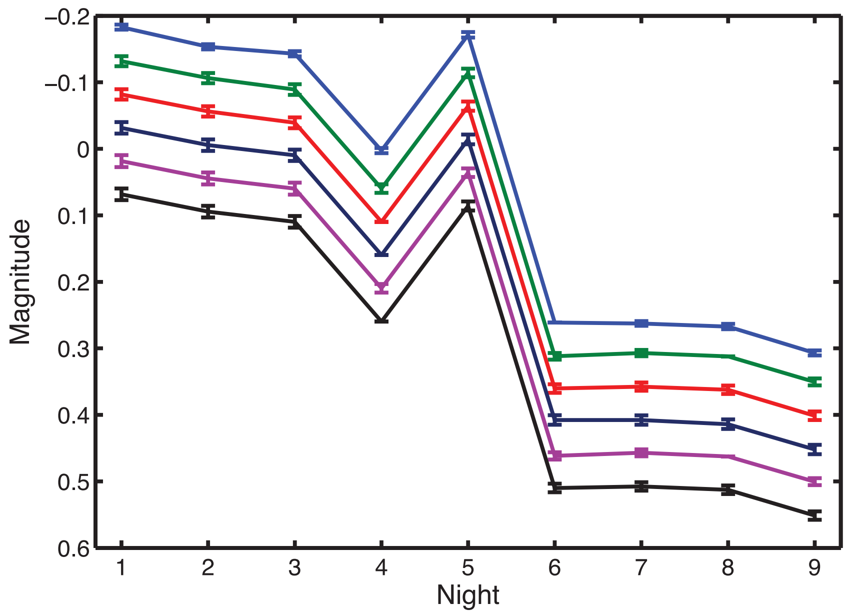

Figure 7. Six different sets of estimates for the zero-points μ_r. Each solution has been arbitrarily shifted vertically for clarity. Note that the full width of the error bars is four standard errors, rather than the usual two standard errors. For each of the solutions there is one zero-width error bar, corresponding to the reference level with index R (e.g. $R = 4$ for solutions plotted third, fourth and last from the top).

In the case of (23), the square root of the maximum variance (i.e. the maximum standard error of any μ_r) is plotted (top panel), while the sum in (21) has been modified to

$$\text{Minimize} \left[\frac{1}{R-1} \sum_{r=1}^{R-1} \text{var}(\mu_r) \right]^{1/2}$$

which is approximately the minimum mean standard error of all the estimated μ_r (middle panel). Note that the units are in millimagnitudes. The squares denote results obtained by starting with the full set of 13 stars; the filled circles are for the subset of nine stars observed at least four times.

Points worthy of note are as follows.

(i) None of the optimal sets of stars includes numbers 5 and 9; a glance at Fig. 5 shows that this makes perfect sense.

(ii) One of the two stars with $M_s = 2$ and both with $M_s = 3$ are included in the subsets selected by criteria (21) and (22), starting from the full-data complement. This shows that even stars with quite small numbers of measurements may be useful in determining the $\widehat{\mu}_r$.

(iii) The standard errors of the μ_r are evidently quite small. This and the similarity of the six different sets of estimates (corresponding to the six minima in Fig. 6) are confirmed by Fig. 7, which shows the optimal estimates with ± 2 standard error bars.

(iv) Differences between solutions are made more explicit in Fig. 8, in which each individual point in Fig. 7 has been adjusted by subtraction of means taken over all six solutions for a given r, and the mean over all r for a given solution. The odd one out, plotted as dashes, corresponds to the criterion (23) subset of stars, selected

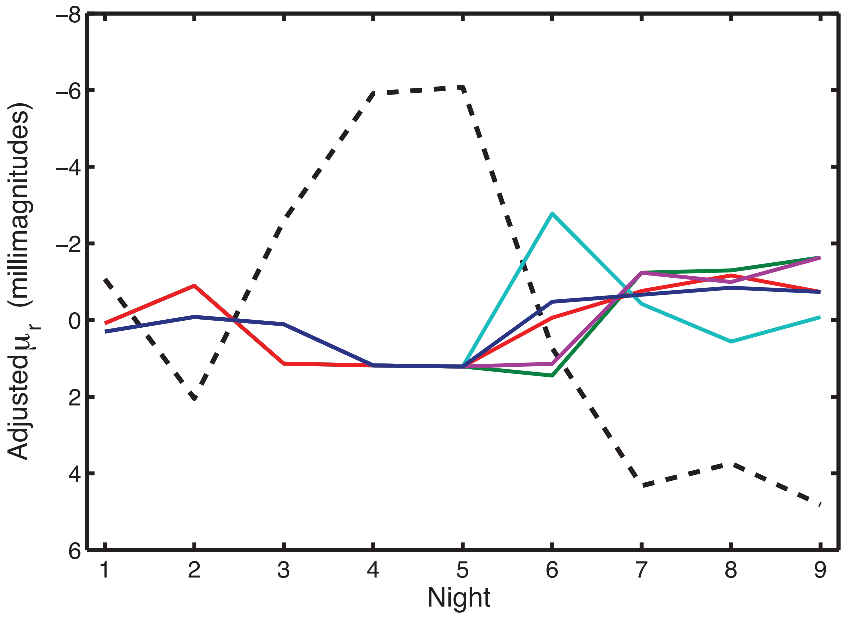

Figure 8. A different view of some of the information in Fig. 7: adjustment has been made for the large night-to-night level changes, and the arbitrariness of the mean value, for any given solution. The dashed line corresponds to the lower of the two minima in the top panel of Fig. 6.

Table 1. Estimated σ_η (in millimagnitudes) for the 2M 1155−3727 *I*-band observations, for the various optimal collections of calibrating stars (columns 4–6, 8–10). Also shown are the $\widehat{\sigma}_\eta$ for the full data set (column 3) and all stars with at least four measurements (column 7).

		Starting from full data				Only $M_s \geq 4$ data			
Star number	$\sigma_{ij}/\sqrt{n_{ij}}$	All	(23)	(21)	(22)	All	(23)	(21)	(22)
1	0.16	0.0	–	0.0	0.0	–	–	–	–
2	0.25	0.0	–	0.0	0.0	–	–	–	–
3	0.46	8.1	1.6	7.7	7.5	9.6	6.6	7.7	7.4
4	0.48	7.4	2.0	5.7	6.2	8.7	5.3	5.1	5.7
5	0.81	12.5	–	–	–	11.9	–	–	–
6	0.62	3.9	–	6.6	4.6	2.6	5.1	6.8	4.7
7	0.80	3.4	–	2.1	4.1	0.0	6.1	2.2	4.4
8	0.86	3.4	–	4.2	4.1	0.0	5.0	4.2	4.2
9	0.57	13.5	–	–	–	–	–	–	–
10	0.94	0.0	0.0	0.0	0.0	–	–	–	–
11	1.07	4.2	–	0.0	2.0	0.0	–	0.0	2.8
12	1.24	7.8	0.5	–	7.5	1.9	5.4	–	–
13	1.17	2.0	2.4	0.0	0.0	0.6	–	0.0	1.5

from the full complement of stars (i.e. the minimum at $S = 5$ in the top panel of Fig. 6).

(v) A probable reason for the discrepancy between the dashed line solution in Fig. 8 and the rest can be seen in Table 1, which lists the estimated σ_η for each star in the selected calibration subset. Comparison of the rows in the table reveals that the estimates in the fourth column generally do not agree very well with the rest. In particular, three of the five estimates are substantially smaller than those obtained from other calibration sets of stars. This can be understood by noting, for example, that only two of the stars selected in this solution (numbers 3 and 4 in Fig. 5) are used to estimate μ_r for $r = 1, 2, \ldots, 5$. This could obviously give rise to

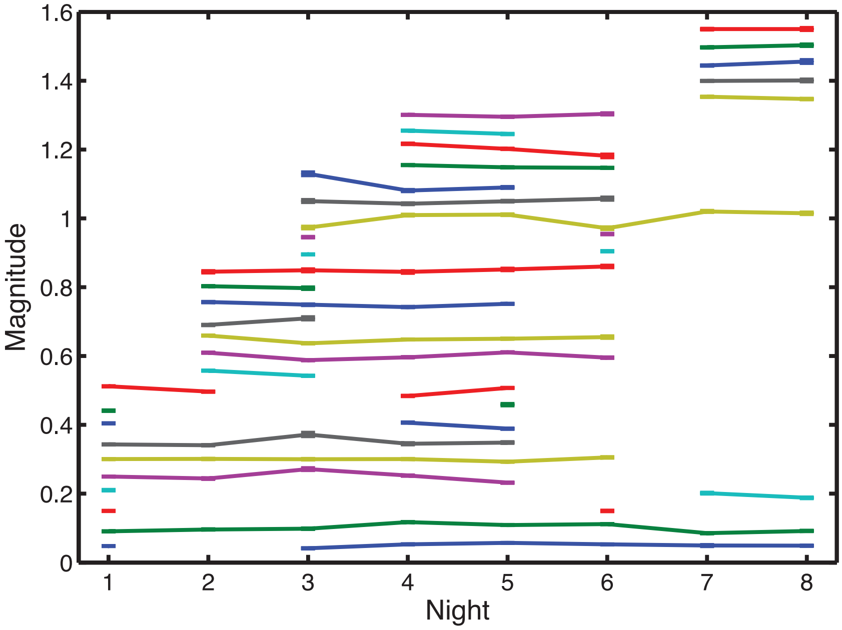

Figure 9. As for Fig. 5, but for the UCD DENIS 1454−6604. Note that only one star was measured on all eight nights.

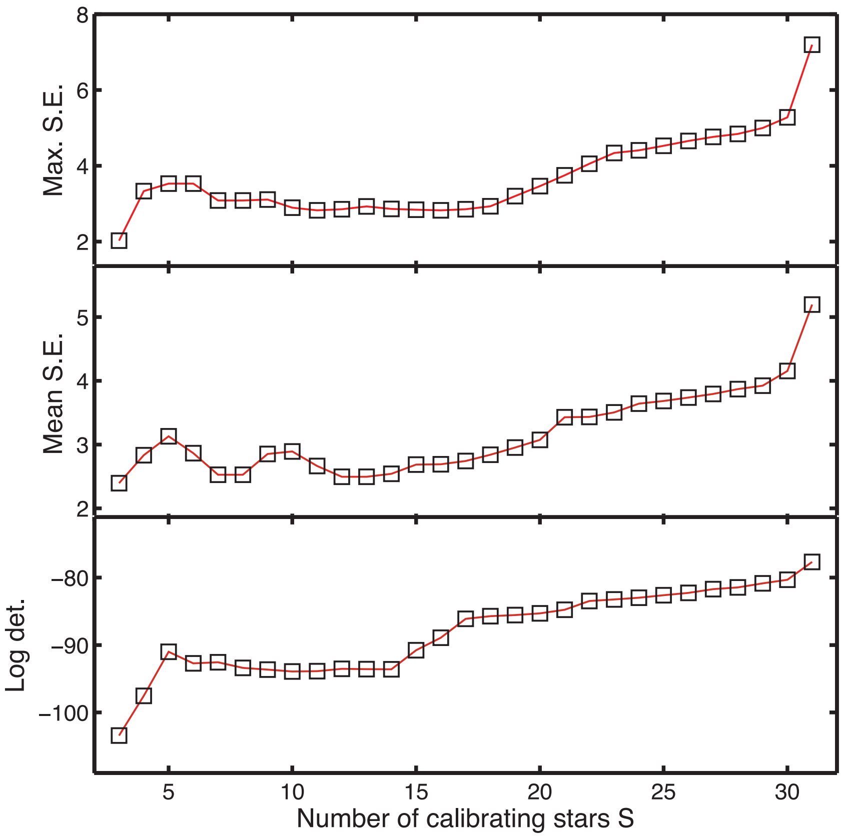

Figure 10. As for Fig. 6, but for DENIS 1454−6604.

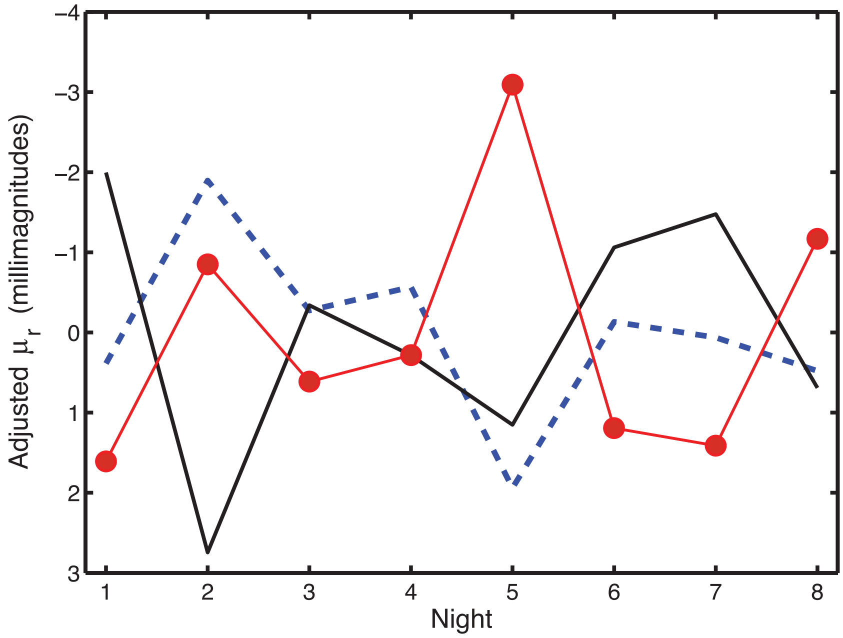

Figure 11. As for Fig. 8, but for DENIS 1454−6604. The solid line, connected dots and dashed line show the $\widehat{\mu}_r$ corresponding, respectively, to criteria (21), (22) and (23).

an erroneous impression of very small variability in these μ_r. What is missing here is some compensation for the *uncertainty* in the estimates of the σ_η: that information could be used to add error bars to the points plotted in Fig. 6, which would most likely reveal that the lower minimum in the top panel of the figure, which is based on data from only three stars, is quite uncertain.

The L3.5 dwarf DENIS-P J1454078−660447 (hereafter DENIS 1454−6604) was monitored on eight nights. The number of time series measurements per night varied from 35 to 140, with $1.7 \le \sigma_{ij} \le 42.1$ mmag. The fraction of missing observations is 0.56, with only one star observed every night (Fig. 9).

The analogue of Fig. 6 is plotted in Fig. 10. Given the remarks in point (v) above, it seems sensible to ignore the minima at $S = 3$, and instead to impose $S \ge 10$, and concentrate on the local minima visible in the range $10 \le S \le 20$. The number of calibrating stars selected by criteria (21)–(23) is then, respectively, 13, 10 and 11. The corresponding three sets of standard errors on the $\widehat{\mu}_r$ lie in the intervals (1.6, 3.3), (2.9, 3.4) and (1.6, 2.8) mmag. The three solutions are also in good agreement – see Fig. 11, which is the analogue of Fig. 8 for 2M 1155−3727. On the other hand, estimates of σ_η for a given star can be widely different – probably reflecting the considerable incompleteness of the data, which leads to several estimates being based on only two data points.

7 CLOSING REMARK

Two important, but (at least to the author) unexpected points emerged in the discussion of the 2M 1155−3727 data in the previous section. The first is that even stars which have only been measured two or three times may be useful in estimation of the μ_r. The second is the importance of quantifying the uncertainties in the points plotted in diagrams such as Fig. 6 – which again requires evaluation of the uncertainties in the $\widehat{\sigma}_\eta$. This may not be straightforward, as a theoretical expression for the covariance matrix of the estimated $\sigma^2_{\eta s}$ will itself contain higher unknown moments of the η_s. The best strategy may be to rely on non-parametric methods, if such can be found.

ACKNOWLEDGEMENTS

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APPENDIX A: ESTIMATION OF $\sigma^2_{\eta s}$, FULL-DATA CASE

The estimated residuals are given by

$$\widehat{e}_{rs} = Y_{rs\bullet} - \widehat{\mu}_r - \widehat{\Delta}_s \,.$$

Consider the expected value

$$E\sum_{r=1}^{R-1}\widehat{e}^2_{rs} = E\sum_{r=1}^{R-1}[(Y_{rs\bullet} - \mu_r - \Delta_s) + (\mu_r + \Delta_s - \widehat{\mu}_r - \widehat{\Delta}_s)]^2$$

$$= \sum_{r=1}^{R-1}[Ee^2_{rs} + \mathrm{cov}(\widehat{\mu}_r + \widehat{\Delta}_s, \widehat{\mu}_r + \widehat{\Delta}_s) - 2\mathrm{cov}(e_{rs}, \widehat{\mu}_r + \widehat{\Delta}_s)]. \quad \text{(A1)}$$

The first term can be written as

$$\sum_{r=1}^{R-1} Ee^2_{rs} = (R-1)\sigma^2_{\eta s} + \sum_{r=1}^{R-1}\frac{\sigma^2_{rs}}{n_{rs}} \quad \text{(A2)}$$

while the last is (cf. equation 15)

$$\mathrm{cov}(e_{rs}, \widehat{\mu}_r + \widehat{\Delta}_s) = \mathrm{cov}(e_{rs}, Y_{r\bullet\bullet} + Y_{\bullet s\bullet} - Y_{\bullet\bullet\bullet})$$

$$= \mathrm{cov}\left(e_{rs}, \frac{1}{S}\sum_k Y_{rk\bullet} + \frac{1}{R}\sum_\ell Y_{\ell s\bullet} - \frac{1}{RS}\sum_k\sum_\ell Y_{k\ell\bullet}\right)$$

$$= \frac{1}{S}\sum_k \mathrm{cov}(e_{rs}, Y_{rk\bullet}) + \frac{1}{R}\sum_\ell \mathrm{cov}(e_{rs}, Y_{\ell s\bullet}) - \frac{1}{RS}\sum_k\sum_\ell \mathrm{cov}(e_{rs}, Y_{k\ell\bullet})$$

$$= \frac{1}{S}\mathrm{var}(e_{rs}) + \frac{1}{R}\mathrm{var}(e_{rs}) - \frac{1}{RS}\mathrm{var}(e_{rs})$$

$$= \frac{R+S-1}{RS}(\sigma^2_{\eta s} + \frac{\sigma^2_{rs}}{n_{rs}}). \quad \text{(A3)}$$

Substituting (A2) and (A3) into (A1), and using (18),

$$E\sum_{r=1}^{R-1}\widehat{e}^2_{rs} = \frac{(R-1)^2(S-2)}{RS}\sigma^2_{\eta s} + \frac{(R-1)^2}{RS^2}\sum_{k=1}^{S}\sigma^2_{\eta k} + \frac{(R-2)(S-2)}{RS}\sum_{r=1}^{R-1}\frac{\sigma^2_{rs}}{n_{rs}}$$

$$+\frac{(R-1)(S-2)}{R^2S}\sum_{r=1}^{R}\frac{\sigma^2_{rs}}{n_{rs}} + \frac{R-2}{RS^2}\sum_{r=1}^{R-1}\sum_{k=1}^{S}\frac{\sigma^2_{rk}}{n_{rk}} + \frac{R-1}{R^2S^2}\sum_{r=1}^{R}\sum_{k=1}^{S}\sigma^2_{rk}/n_{rk}. \quad \text{(A4)}$$

Summing over all the stars,

$$E\sum_{s=1}^{S}\sum_{r=1}^{R-1}\widehat{e}^2_{rs} = \frac{(R-1)^2(S-1)}{RS}\sum_{s=1}^{S}\sigma^2_{\eta s} + \frac{(R-2)(S-1)}{RS}\sum_{r=1}^{R-1}\sum_{s=1}^{S}\frac{\sigma^2_{rs}}{n_{rs}} + \frac{(R-1)(S-1)}{R^2S}\sum_{r=1}^{R}\sum_{s=1}^{S}\frac{\sigma^2_{rs}}{n_{rs}}. \quad \text{(A5)}$$

Equation (A5) provides an estimator for the sum over the $\sigma^2_{\eta s}$:

$$\sum_{s=1}^{S}\widehat{\sigma}^2_{\eta s} = \frac{RS}{(R-1)^2(S-1)}\sum_{s=1}^{S}\sum_{r=1}^{R-1}\widehat{e}^2_{rs} - \frac{(R-2)}{(R-1)^2}\sum_{r=1}^{R-1}\sum_{s=1}^{S}\frac{\sigma^2_{rs}}{n_{rs}} - \frac{1}{R(R-1)}\sum_{r=1}^{R}\sum_{s=1}^{S}\frac{\sigma^2_{rs}}{n_{rs}} \quad \text{(A6)}$$

which can be substituted into (A4) to give

$$\widehat{\sigma}^2_{\eta s} = \frac{RS}{(R-1)^2(S-2)}\sum_{r=1}^{R-1}\widehat{e}^2_{rs} - \frac{R}{(R-1)^2(S-1)(S-2)}\sum_{s=1}^{S}\sum_{r=1}^{R-1}\widehat{e}^2_{rs} - \frac{R-2}{(R-1)^2}\sum_{r=1}^{R-1}\frac{\sigma^2_{rs}}{n_{rs}} - \frac{1}{R(R-1)}\sum_{r=1}^{R}\frac{\sigma^2_{rs}}{n_{rs}}. \quad \text{(A7)}$$

In the case when $\sigma_{\eta 1} = \sigma_{\eta 2} = \cdots = \sigma_{\eta S} \equiv \sigma_\eta$,

$$\widehat{\sigma}^2_{\eta} = \frac{R}{(R-1)^2(S-1)}\sum_{s=1}^{S}\sum_{r=1}^{R-1}\widehat{e}^2_{rs} - \frac{R-2}{(R-1)^2S}\sum_{r=1}^{R-1}\sum_{s=1}^{S}\frac{\sigma^2_{rs}}{n_{rs}} - \frac{1}{R(R-1)S}\sum_{r=1}^{R}\sum_{s=1}^{S}\frac{\sigma^2_{rs}}{n_{rs}}. \quad \text{(A8)}$$

APPENDIX B: ESTIMATION OF $\sigma^2_{\eta s}$, PARTIAL-DATA CASE

The summation in (A1) and (A2) now only extends over the nights during which star s was measured:

$$E\sum_{r\,\in\,\mathcal{R}'_s}\widehat{e}^2_{rs}=\sum_{r\,\in\,\mathcal{R}'_s}[Ee^2_{rs}+\mathrm{cov}(\widehat{\mu}_r+\widehat{\Delta}_s,\widehat{\mu}_r+\widehat{\Delta}_s)-2\mathrm{cov}(e_{rs},\widehat{\mu}_r+\widehat{\Delta}_s)] \tag{B1}$$

and

$$\sum_{r\,\in\,\mathcal{R}'_s}Ee^2_{rs}=M'_s\sigma^2_{\eta s}+\sum_{r\,\in\,\mathcal{R}'_s}\frac{\sigma^2_{rs}}{n_{rs}}, \tag{B2}$$

where $\mathcal{R}'_s$ is the set of nights during which star s was observed, *excluding* night R; M'_s is the cardinality of $\mathcal{R}_s$. Proceeding from (12), it is not difficult to prove that

$$\begin{aligned}\mathrm{cov}(e_{rs},\widehat{\mu}_r+\widehat{\Delta}_s)&=\mathrm{cov}(e_{rs},g_{rr}U_r)+\mathrm{cov}(e_{rs},g_{r,R-1+s}V_s)+\mathrm{cov}(e_{rs},g_{R-1+s,r}U_r)+\mathrm{cov}(e_{rs},g_{R-1+s,R-1+s}V_s)\\&=\mathrm{var}(e_{rs})(g_{rr}+g_{r,R-1+s}+g_{R-1+s,r}+g_{R-1+s,R-1+s}),\end{aligned}$$

where the notation $g_{ij}\equiv \boldsymbol{G}^{-1}_{ij}$, the (i,j) entry in the matrix $\mathbf{G}^{-1}$, and U_r and V_s are defined in (8). Summing,

$$\begin{aligned}\sum_{r\,\in\,\mathcal{R}'_s}\mathrm{cov}(e_{rs},\widehat{\mu}_r+\widehat{\Delta}_s)&=\sigma^2_{\eta s}\sum_{r=1}^{R-1}(g_{rr}+g_{r,R-1+s}+g_{R-1+s,r}+g_{R-1+s,R-1+s})A(r,s)\\&\quad+\sum_{r=1}^{R-1}(g_{rr}+g_{r,R-1+s}+g_{R-1+s,r}+g_{R-1+s,R-1+s})A(r,s)\frac{\sigma^2_{rs}}{n_{rs}}\\&\equiv\alpha_0(s)\sigma^2_{\eta s}+\beta_0(s).\end{aligned} \tag{B3}$$

Also, in terms of entries in the covariance matrix $\mathbf{C}$ in (16),

$$\mathrm{cov}(\widehat{\mu}_r+\widehat{\Delta}_s,\widehat{\mu}_r+\widehat{\Delta}_s)=\boldsymbol{C}_{rr}+2\boldsymbol{C}_{r,R-1+s}+\boldsymbol{C}_{R-1+s,R-1+s}. \tag{B4}$$

After some algebra,

$$\begin{aligned}\boldsymbol{C}_{rr}&=\sum_{j=1}^{R-1}\sum_{k=1}^{S}(g_{rj}+g_{r,R-1+k})^2A(j,k)\mathrm{var}(e_{jk})+\sum_{k=1}^{S}g^2_{r,R-1+k}A(R,k)\mathrm{var}(e_{Rk})\\ \boldsymbol{C}_{R-1+s,R-1+s}&=\sum_{j=1}^{R-1}\sum_{k=1}^{S}(g_{R-1+s,j}+g_{R-1+s,R-1+k})^2A(j,k)\mathrm{var}(e_{jk})+\sum_{k=1}^{S}g^2_{R-1+s,k}A(R,k)\mathrm{var}(e_{Rk})\\ \boldsymbol{C}_{r,R-1+s}&=\sum_{j=1}^{R-1}\sum_{k=1}^{S}(g_{r,j}+g_{r,R-1+k})(g_{j,R-1+s}+g_{R-1+k,R-1+s})A(j,k)\mathrm{var}(e_{jk})\\&\quad+\sum_{k=1}^{S}g_{r,R-1+k}g_{R-1+k,R-1+s}A(R,k)\mathrm{var}(e_{Rk}).\end{aligned} \tag{B5}$$

Summing over r,

$$\begin{aligned}\sum_{r\,\in\,\mathcal{R}'_s}\boldsymbol{C}_{rr}&=\sum_{k=1}^{S}\sigma^2_{\eta k}\left[\sum_{r=1}^{R-1}A(r,s)\sum_{j=1}^{R-1}(g_{rj}+g_{r,R-1+k})^2A(j,k)+\sum_{r=1}^{R-1}g^2_{r,R-1+k}A(R,k)\right]\\&\quad+\sum_{k=1}^{S}\sum_{r=1}^{R-1}A(r,s)\left[g^2_{r,R-1+k}A(R,k)\frac{\sigma^2_{Rk}}{n_{Rk}}+\sum_{j=1}^{R-1}(g_{rj}+g_{r,R-1+k})^2A(j,k)\frac{\sigma^2_{jk}}{n_{jk}}\right]\\&\equiv\sum_{k=1}^{S}\alpha_1(s,k)\sigma^2_{\eta k}+\beta_1(s)\end{aligned} \tag{B6}$$

$$\sum_{r\,\in\,\mathcal{R}'_s} \boldsymbol{C}_{R-1+s,R-1+s} = M'_s \sum_{k=1}^{S} \sigma^2_{\eta k} \left[g^2_{s,R-1+k} A(R,k) + \sum_{j=1}^{R-1} (g_{sj} + g_{s,R-1+k})^2 A(j,k) \right]$$

$$+ M'_s \sum_{k=1}^{S} \left[g^2_{s,R-1+k} A(R,k) \frac{\sigma^2_{Rk}}{n_{Rk}} + \sum_{j=1}^{R-1} (g_{sj} + g_{s,R-1+k})^2 A(j,k) \frac{\sigma^2_{jk}}{n_{jk}} \right]$$

$$\equiv \sum_{k=1}^{S} \alpha_2(s,k) \sigma^2_{\eta k} + \beta_2(s) \qquad \text{(B7)}$$

$$\sum_{r\,\in\,\mathcal{R}'_s} \boldsymbol{C}_{r,R-1+s} = \sum_{k=1}^{S} \sigma^2_{\eta k} \left[\sum_{r=1}^{R-1} A(r,s) \sum_{j=1}^{R-1} (g_{r,j} + g_{r,R-1+k})(g_{j,R-1+s} + g_{R-1+k,R-1+s}) A(j,k) \right.$$

$$\left. + \sum_{r=1}^{R-1} A(r,s) g_{r,R-1+k} g_{R-1+k,R-1+s} A(R,k) \right] + \sum_{k=1}^{S} \sum_{r=1}^{R-1} A(r,s) \left[g^2_{r,R-1+k} A(R,k) \frac{\sigma^2_{Rk}}{n_{Rk}} \right.$$

$$\left. + \sum_{j=1}^{R-1} (g_{r,j} + g_{r,R-1+k})(g_{j,R-1+s} + g_{R-1+k,R-1+s}) A(j,k) \frac{\sigma^2_{jk}}{n_{jk}} \right]$$

$$\equiv \sum_{k=1}^{S} \alpha_3(s,k) \sigma^2_{\eta k} + \beta_3(s). \qquad \text{(B8)}$$

Substituting (B2)–(B8) into (B1),

$$E \sum_{r\,\in\,\mathcal{R}'_s} \widehat{e}^2_{rs} = \left[M'_s - 2\alpha_0(s) \right] \sigma^2_{\eta s} + \sum_{k=1}^{S} \left[\alpha_1(k) + \alpha_2(s,k) + 2\alpha_3(s,k) \right] \sigma^2_{\eta k}$$

$$+ \left[-2\beta_0(s) + \beta_1(s) + \beta_2(s) + 2\beta_3(s) + \sum_{r=1}^{R-1} A(r,s) \frac{\sigma^2_{rs}}{n_{rs}} \right]. \qquad \text{(B9)}$$

Replacing the expected value on the left-hand side of (B9) by its sample value, the estimator

$$\widehat{\boldsymbol{\sigma}}^2_{\eta} = \boldsymbol{Q}^{-1} \boldsymbol{q}$$

$$Q(s,k) = [M'_s - 2\alpha_0(s)]\delta_{ks} + \alpha_1(s,k) + \alpha_2(s,k) + 2\alpha_3(s,k)$$

$$q(s) = 2\beta_0(s) - \beta_1(s) - \beta_2(s) - 2\beta_3(s) + \sum_{r=1}^{R-1} A(r,s) \left[\widehat{e}^2_{rs} - \frac{\sigma^2_{rs}}{n_{rs}} \right] \qquad \text{(B10)}$$

is obtained; $\boldsymbol{\sigma}^2_{\eta}$ is a column vector with entries $\sigma^2_{\eta s}$ ($s = 1, 2, \ldots, S$).

If it can be assumed that all $\sigma_{\eta s}$ are equal, then the common variance σ^2_{η} can be estimated from

$$\widehat{\sigma}^2_{\eta} = \frac{\sum_{r=1}^{R-1} \sum_{s=1}^{S} A(r,s) \left(\widehat{e}^2_{rs} - \sigma^2_{rs}/n_{rs} \right) + \sum_{s=1}^{S} (2\beta_0 - \beta_1 - \beta_2 - 2\beta_3)}{\sum_{s=1}^{S} (M'_s - 2\alpha_0) + \sum_{s=1}^{S} \sum_{k=1}^{S} (\alpha_1 + \alpha_2 + 2\alpha_3)}. \qquad \text{(B11)}$$

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