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The analysis of indexed astronomical time series – X. Significance testing of $O - C$ data

Chris Koen★

Department of Statistics, University of the Western Cape, Private Bag X17, Bellville, 7535 Cape, South Africa

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ABSTRACT

It is assumed that $O - C$ ('observed minus calculated') values of periodic variable stars are determined by three processes, namely measurement errors, random cycle-to-cycle jitter in the period, and possibly long-term changes in the mean period. By modelling the latter as a random walk, the covariances of all $O - C$ values can be calculated. The covariances can then be used to estimate unknown model parameters, and to choose between alternative models. Pseudo-residuals which could be used in model fit assessment are also defined. The theory is illustrated by four applications to spotted stars in eclipsing binaries.

Key words: methods: data analysis – methods: statistical – stars: variables: other.

1 INTRODUCTION

Koen (2005) recently presented a very brief review of the literature on rapid, non-evolutionary changes in the periods of variable stars. Examples included the effects of small abundance gradient changes on the the periods of pulsating stars, and of the 'Applegate effect' (e.g. Applegate 1992) on binary periods. Of course, convection is a well-known stochastic process, which has also been mentioned as a possible driver of small period changes in variable stars (e.g. Deasy & Wayman 1985; Wolff et al. 2002).

The type of period fluctuation described above could manifest itself in at least two different ways: as cycle-to-cycle 'jitter' in the period, or, if the effects are cumulative, as a systematic change in the period. The latter would not necessarily resemble the usually assumed sinusoids or low-order polynomials, but would rather be stochastic trends. This line of thinking was explored in some detail by Koen (1996); he found that good model fits to many $O - C$ ('observed minus calculated') data sets could be obtained with purely stochastic models.

The present paper is concerned with test statistics which can be used to decide which phenomena (measurement error, period jitter and/or systematic period changes) best explain a set of $O - C$ observations. This contrasts with the typical approach in which regression techniques are used to try to ascertain the precise time evolution of the period.

The statistical model formulation is dealt with in Section 2; the test procedure is covered in Section 3; the model fit evaluation is discussed in Section 4; analysis of a few data sets is presented in Section 5; and conclusions are given in Section 6. Some of the necessary algebra is presented in the Appendix.

★E-mail: ckoen@uwc.ac.za

2 THE STATISTICAL MODEL

The 'instantaneous' period at epoch j is assumed to be composed of two components: a slowly, and relatively smoothly, varying part $\mu(j)$ (which could be a constant); and a stochastic part which fluctuates randomly from cycle to cycle. The random fluctuation at epoch j will be modelled by the zero mean uncorrelated random variables η_j, which are assumed to have constant variance σ_η^2. Such fluctuations have been referred to in the literature as 'period jitter' or 'intrinsic period scatter'.

For most types of variable star σ_η^2 may be very small, but this does not necessarily imply that its effects on $O - C$ diagrams will be negligible. The point is easily demonstrated using simulated data. Assuming a measurement error standard deviation of 5×10^{-4} of the period, and 100 observations uniformly distributed over 10 000 cycles, the top panel of Fig. 1 shows the flat $O - C$ for a constant period and $\sigma_\eta = 0$. In the next panel period jitter with $\sigma_\eta = 2 \times 10^{-5}$ of the period has been added: the result (for this particular simulation) is an apparent sinusoidal modulation of the $O - C$. Clearly, since the period jitter is at the level of 4 per cent of the measurement error, it would be undetectable from individual cycles of variation of the star.

The third and last panels of Fig. 1 respectively demonstrate the effects of increasing σ_η and decreasing the level of measurement error. Furthermore, since the effects of period jitter on the $O - C$ values are cumulative, a longer baseline of observations would generally give a larger apparent period change.

There are therefore at least two reasons for including intrinsic random variability in the period in the formulation. The first is the fact that the effects of non-zero σ_η could mimic those of systematic changes in the mean period [i.e. in $\mu(t)$], or bias estimates of $\mu(t)$ if both systematic and random changes are present. The second reason is that it may be possible to learn something about the level of period jitter despite the fact that it is inaccessible to direct measurement.

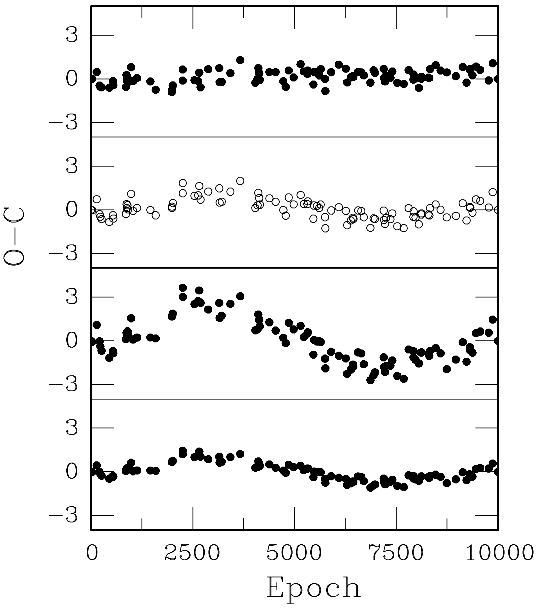

Figure 1. Simulated $O - C$ diagrams for a constant mean period. From top to bottom ($\sigma_e = 5 \times 10^{-4}, \sigma_\eta = 0$), ($\sigma_e = 5 \times 10^{-4}, \sigma_\eta = 2 \times 10^{-5}$), ($\sigma_e = 5 \times 10^{-4}, \sigma_\eta = 5 \times 10^{-5}$), ($\sigma_e = 2 \times 10^{-4}, \sigma_\eta = 2 \times 10^{-5}$).

This could further understanding of the levels of stability of periodic variability.

A very general model for the relatively smooth long-term variations in the period denoted by $\mu(t)$ is

$$\mu(t) = \mu_0 + \sum_{j=1}^{t} \xi_j, \tag{1}$$

where the ξ_j are zero mean uncorrelated random variables with variance σ_ξ^2. Simulation examples of the efficacy of modelling smooth functions (typically resembling low-order polynomials) by the random walk model (1) can be found in Koen (1996); in a more general context see e.g. Harvey (1989).

The full model for the instantaneous period P_j is then

$$P_j = \mu_j + \eta_j = \mu_0 + \sum_{k=1}^{j} \xi_k + \eta_j \ . \tag{2}$$

Assume that there are $K + 2$ observations (timings) available, t_0, $t_1, t_2, \cdots, t_K, t_{K+1}$. Let the corresponding cumulative cycle numbers be $N_0 = 0$, N_1, N_2, $N_3, \cdots, N_{K+1} \equiv N$, i.e. there are $n_j = N_j - N_{j-1}$ cycles between t_j and t_{j-1}. The jth $O - C$ value is then

$$Z_j \equiv (O - C)_j = t_j - t_0 - N_j \overline{P}, \qquad j = 1, 2, \cdots, K \ . \tag{3}$$

In (3) $\overline{P}$ is the mean period over the $N = N_{K+1}$ cycles:

$$\overline{P} = \frac{t_{K+1} - t_0}{N} \ . \tag{4}$$

Note that given the definition (3), $(O - C)_0 \equiv (O - C)_{K+1} \equiv 0$.

The model specification is completed by noting that the t_j are subject to zero mean measurement errors e_j with constant variance σ_e^2:

$$t_j = T_j + e_j, \qquad j = 0, 1, 2, \cdots, K + 1, \tag{5}$$

where the T_j are the true, error-free values of the timings.

The expected values of all the Z_j are zero, while the entries in their covariance matrix $\boldsymbol{\Sigma}$ are derived in Appendix A.

Consider first the case where there is no long-term variation in the period, i.e. $\mu(t) = \mu_0$ in (1). Setting $\sigma_\xi = 0$ in (A6) and (A7) then leaves terms in σ_e^2 and σ_η^2 only. Examination of the two equations shows that these terms are of order unity and N_j respectively. The implication is that the effects of period jitter, as modelled by the η_j, become comparable to those of measurement error in the $O - C$ observations for cumulative cycle numbers

$$N \gtrsim \left(\frac{\sigma_e}{\sigma_\eta}\right)^2 . \tag{6}$$

To illustrate, in the example in the second panel of Fig. 1, $\sigma_e/\sigma_\eta = 25$, hence the the effects of random intrinsic variations in the period could start making an impact for N in excess of about 625.

By contrast, if the systematic period change can be modelled as in (1), its long-term effects grow much more rapidly ($\propto N^3$) with increasing cycle count.

3 TESTING

Within the framework of the model formulated in Section 2, there are four possibilities:

M1: the mean period is constant and there are no random cycle-to-cycle period variations ($\sigma_\eta = 0, \sigma_\xi = 0$);

M2: the mean period is constant but there are random cycle-to-cycle period variations ($\sigma_\eta \neq 0, \sigma_\xi = 0$);

M3: the mean period is variable but there are no random cycle-to-cycle period variations ($\sigma_\eta = 0, \sigma_\xi \neq 0$);

M4: the mean period is variable and there are random cycle-to-cycle period variations ($\sigma_\eta \neq 0, \sigma_\xi \neq 0$).

One way of proceeding is to compare the four models by pairwise hypothesis testing. A possible test statistic is then the standard likelihood ratio

$$\Lambda_{ij} = 2(\mathcal{L}_i - \mathcal{L}_j), \tag{7}$$

where the Gaussian log likelihood is given by

$$\mathcal{L} = -\frac{1}{2}\left(K \log 2\pi + \log|\boldsymbol{\Sigma}| + \mathbf{Z}^{\mathrm{t}}\boldsymbol{\Sigma}^{-1}\mathbf{Z}\right). \tag{8}$$

In (8), $\mathbf{Z}$ is the vector consisting of the K observed $O - C$ values, and $|\boldsymbol{\Sigma}|$ is the determinant of the covariance matrix $\boldsymbol{\Sigma}$. The likelihoods $\mathcal{L}_i$ and $\mathcal{L}_j$ in (7) are calculated by setting the appropriate variances (σ_η^2 and/or σ_ξ^2) in $\boldsymbol{\Sigma}$ equal to zero, and then maximizing with respect to σ_e^2 and the remaining variances (if any).

Aside from the erosion of significance levels due to multiple hypothesis testing, the Λ_{ij} have non-standard distributions due to the fact that the hypotheses involve parameter values that lie on the boundaries of their domains (e.g. Andrews 2001). These and other complications are easily avoided by resorting to the use of model selection statistics: for our purposes useful forms are the Akaike and Bayes information criteria, defined by

$$AIC = -2\log\mathcal{L} + 2p + \frac{2p(p+1)}{K - p - 1},$$
$$BIC = -2\log\mathcal{L} + p\log K \tag{9}$$

(e.g. Harvey 1989; Burnham & Anderson 2004). The symbol p denotes the number of model parameters ($p = 1, 2, 2, 3$ for models M1–M4 respectively). The two criteria essentially contrast a measure of how well the models fit the data (the likelihood term) with a penalty term for the number of fitted parameters ($2p$ and $p \log K$ respectively). The last term in the expression for the AIC is a small sample bias correction (e.g. Burnham & Anderson 2004). The maximum value of the likelihood is calculated for each of the four models, the results substituted into (9), and the model giving the minimum value of the information criteria selected.

Let $AIC(i)$ be the Akaike information criterion for model Mi, and $AIC_{\min}$ the minimum value of $AIC(i)$ ($i = 1, 2, 3, 4$). Then

$$p_A(i) = \frac{\exp\{-[AIC(i) - AIC_{\min}]/2\}}{\sum_j \exp\{-[AIC(j) - AIC_{\min}]/2\}} \tag{10}$$

can be interpreted as the probability that model i is appropriate (Burnham & Anderson 2004). The case of the Bayes information criterion is analogous: probabilities based on the $BIC(i)$ will be denoted by $p_B\ (i)$.

4 FIT EVALUATION

Once the optimal model has been selected, it is possible to investigate its fit to the data, similarly to the evaluation of regression residuals. The covariance matrix of the $O - C$ values is

$$\mathrm{cov}(\boldsymbol{Z}, \boldsymbol{Z}) = \mathrm{E}\boldsymbol{Z}\boldsymbol{Z}^{\mathrm{t}} = \boldsymbol{\Sigma} \tag{11}$$

(E is the expectation operator). The covariance matrix $\boldsymbol{\Sigma}$ is positive-definite and symmetric, and can therefore be written as

$$\boldsymbol{\Sigma} = \mathbf{L}\mathbf{L}^{\mathrm{t}} \tag{12}$$

where $\mathbf{L}$ is lower triangular. This 'Cholesky decomposition' is discussed in, for example, Healy (1986). Defining

$$\mathbf{u} = \mathbf{L}^{-1}\boldsymbol{Z}, \tag{13}$$

it follows from (11)–(13) that

$$\begin{aligned} \mathbf{S}_u = \mathrm{cov}(\mathbf{u}, \mathbf{u}) &= \mathrm{E}\left\{\mathbf{L}^{-1}\boldsymbol{Z}\boldsymbol{Z}^{\mathrm{t}}\left[\mathbf{L}^{-1}\right]^{\mathrm{t}}\right\} \\ &= \mathbf{L}^{-1}\left[\mathrm{E}\boldsymbol{Z}\boldsymbol{Z}^{\mathrm{t}}\right]\left[\mathbf{L}^{-1}\right]^{\mathrm{t}} \\ &= \mathbf{L}^{-1}\boldsymbol{\Sigma}\left[\mathbf{L}^{-1}\right]^{\mathrm{t}} \\ &= \mathbf{L}^{-1}\left[\mathbf{L}\mathbf{L}^{\mathrm{t}}\right]\left[\mathbf{L}^{\mathrm{t}}\right]^{-1} \\ &= \mathbf{I}, \end{aligned} \tag{14}$$

where $\mathbf{I}$ is the identity matrix.

A critical point is that (11), and hence all the equations following it, only applies if the model is appropriate: covariance matrices calculated on the basis of incorrect models will *not* reflect the covariances of the components of $\boldsymbol{Z}$. The implication is that the covariance matrix $\mathbf{S}_u$ of $\mathbf{u}$ is an identity matrix only if $\boldsymbol{\Sigma}$ is calculated for the correct model. Clearly, the components of $\mathbf{u}$ can be interpreted as model residuals; these have zero mean and unit variance, and are uncorrelated only if they are associated with the correct model. The u_j will be referred to as 'pseudo-residuals' in what follows, since they are not derived from a detailed model fit as in e.g. Koen (1996).

Of course, if none of M1–M4 is appropriate, then in general $\mathbf{S}_u \neq \mathbf{I}$.

A few fairly elementary procedures can be applied to the pseudo-residuals in order to see whether these are indeed approximately identically distributed and uncorrelated. A simple plot of u_j against j is often sufficient; it can be supplemented by the autocorrelation function (acf) $r(k)$ of the u_j:

$$r(k) = \frac{1}{K}\sum_{j=1}^{K-k} u_j u_{j+k}, \qquad k = 1, 2, \cdots, J, \tag{15}$$

usually calculated at lags k up to $J = 10$ or 15. A rough 5 per cent significance level for individual values of $|r(k)|$ is $2/\sqrt{K}$. An overall significance level for the entire acf can be found from the portmanteau statistic

$$Q = K\sum_{k=1}^{J}[r(k)]^2 \tag{16}$$

which is approximately chi-squared distributed with $J - p$ degrees of freedom, where p is the number of model parameters (see e.g. Chatfield 2004).

5 SOME RESULTS

As examples, the four data sets discussed by Sowell et al. (2001) are re-analysed. The timings are eclipse minima of four spotted stars in binary systems.

(i) **UV Psc.** The $O - C$ diagram of the photoelectric data is plotted in Fig. 2, using $P = 0.861\,047\,53$ d. The results of fitting the four models, with and without the outlier visible in Fig. 2, are given in Table 1. The best model, according to both *AIC* and *BIC*, is M2 (period jitter only – no systematic change in period).

The solutions for σ_ξ in M4 are so small that the model essentially reduces to M2. By implication M4 should be properly be excluded from the calculation of the p_A and p_B, making M2 even more dominant.

The pseudo-residuals are plotted in Fig. 3; all three sets are statistically uncorrelated. (Due to the fact that $\sigma_\xi \approx 0$ in M4, its pseudo-residuals are very close to those of M2, and are therefore not plotted.)

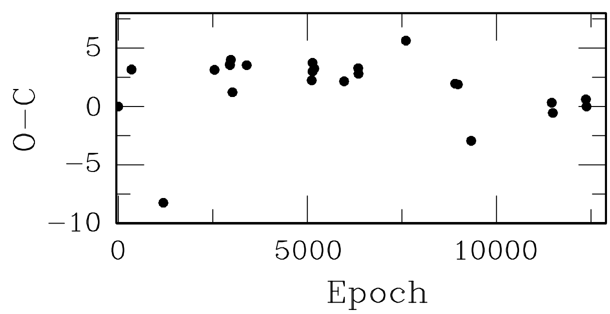

Figure 2. The $O - C$ diagram of the photoelectric photometry of UV Psc, as tabulated by Sowell et al. (2001). The scale on the vertical axis is in units of 0.001 d.

Table 1. Results of fitting each of the four models to the UV Psc data. The estimated standard deviations are given in units of days. The last two columns are the Akaike and Bayes model probabilities [see equation (10)].

Model	σ_e	σ_η	σ_ξ	p_A	p_B
		All data			
M1	2.9×10^{-3}	0	0	0.00	0.00
M2	3.8×10^{-4}	1.7×10^{-4}	0	0.79	0.82
M3	2.6×10^{-3}	0	1.5×10^{-8}	0.00	0.00
M4	3.8×10^{-4}	1.7×10^{-4}	1.0×10^{-14}	0.21	0.18
		Excluding the outlier at Epoch 1194			
M1	1.7×10^{-3}	0	0	0.13	0.17
M2	6.7×10^{-4}	9.5×10^{-5}	0	0.55	0.54
M3	1.5×10^{-3}	0	1.0×10^{-8}	0.18	0.17
M4	6.7×10^{-4}	9.5×10^{-5}	3.1×10^{-15}	0.14	0.12

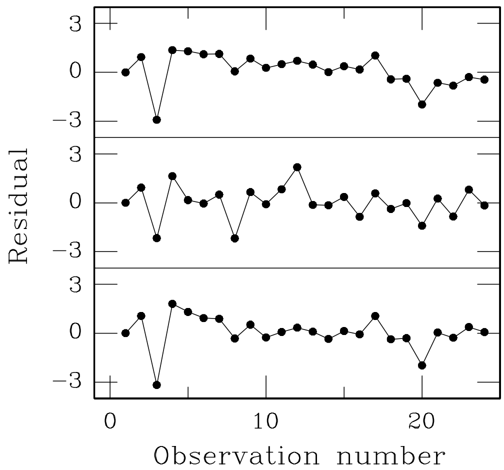

Figure 3. The pseudo-residuals of the UV Psc data, after fitting models M1 (top panel), M2 (middle panel) and M3 (bottom panel). The model M4 pseudo-residuals are indistinguishable from those of the M2 model.

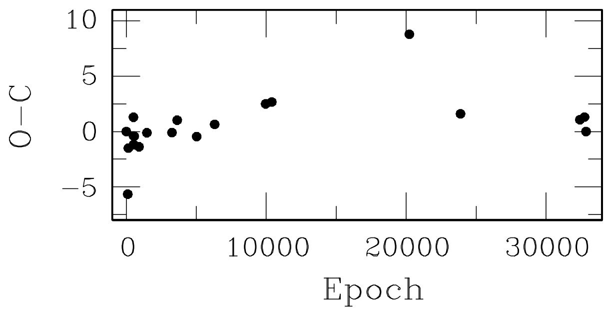

Figure 4. The $O-C$ diagram based on all observations of YY Gem, as tabulated by Sowell et al. (2001). The scale on the vertical axis is in units of 0.001 d.

Table 2. Results of fitting each of the four models to the YY Gem data. The estimated standard deviations are given in units of days. The last two columns are the Akaike and Bayes model probabilities [see equation (10)].

Model	σ_e	σ_η	σ_ξ	p_A	p_B
		All data			
M1	2.5×10^{-3}	0	0	0.04	0.05
M2	1.6×10^{-4}	4.4×10^{-5}	0	0.39	0.39
M3	1.8×10^{-3}	0	5.3×10^{-9}	0.45	0.45
M4	1.6×10^{-4}	2.9×10^{-5}	4.8×10^{-9}	0.12	0.12

(ii) **YY Gem.** The $O-C$ plot, based on $P = 0.814\,282\,20$ d, is shown in Fig. 4, and the model-fitting results are summarized in Table 2. There is not much to choose between M2 and M3, but M4 is less likely. There are only $N = 20$ data points, so the lack of a clear choice is perhaps not surprising.

The pseudo-residuals of all the models are consistent with being white noise.

(iii) **CG Cyg.** A period of 0.631 143 12 d was used to calculate the $O-C$ values in Fig. 5. Properties of the fitted models are summarized in Table 3: the model with both a systematic period change and intrinsic period scatter is preferred. The pseudo-residuals for models M2–M4 are plotted in Fig. 6, and their acfs are shown in Fig. 7. Somewhat surprisingly, the level of residual autocorrelation in M4 is higher than in M2 and M3: the portmanteau Q statistics are 16.7, 10.1 and 21.6 for M2, M3 and M4 respectively [using $J = 10$ lags in equation (16)]. The Q-value for M4 is highly significant, meaning that there is considerable autocorrelation in the residuals. The result is due to the presence of the large residuals with alternating signs visible in the central section of the residual time series.

(iv) **XY UMa.** Fig. 8 is the $O-C$ diagram, calculated using $P = 0.478\,995\,01$ d. The model selection results in Table 4 are similar to those for UV Psc given in Table 1: M4 is essentially equivalent to

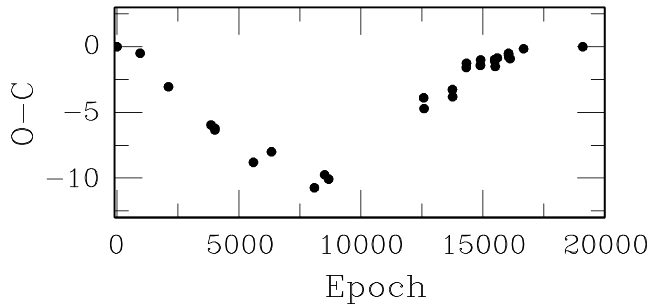

Figure 5. The $O-C$ diagram based on the photoelectric photometry of CG Cyg, as tabulated by Sowell et al. (2001). The scale on the vertical axis is in units of 0.001 d.

Table 3. Results of fitting each of the four models to the CG Cyg data. The estimated standard deviations are given in units of days. The last two columns are the Akaike and Bayes model probabilities [see equation (10)].

Model	σ_e	σ_η	σ_ξ	p_A	p_B
		All data			
M1	2.9×10^{-3}	0	0	0.00	0.00
M2	2.1×10^{-4}	5.2×10^{-5}	0	0.11	0.14
M3	4.7×10^{-4}	0	2.2×10^{-8}	0.13	0.18
M4	2.7×10^{-4}	3.1×10^{-5}	1.9×10^{-8}	0.76	0.67

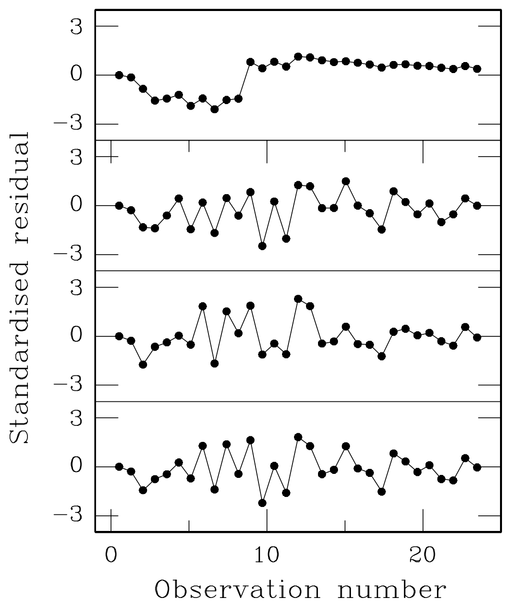

Figure 6. The pseudo-residuals of the CG Cyg data, after fitting models M1–M4 (from top to bottom).

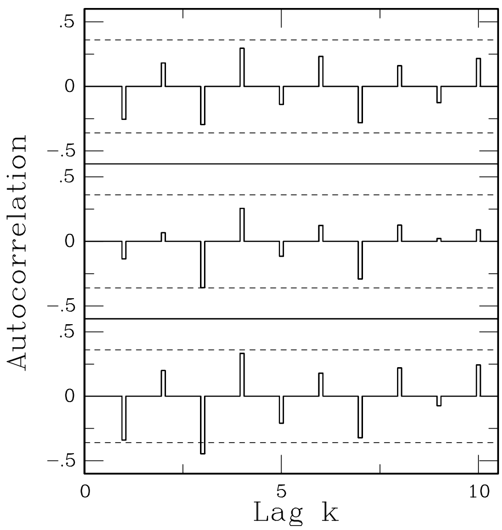

Figure 7. The autocorrelation functions of the pseudo-residuals of models M2–M4 from Fig. 6. The broken lines are the approximate 95 per cent confidence bounds for zero correlation [see the text after equation (15)].

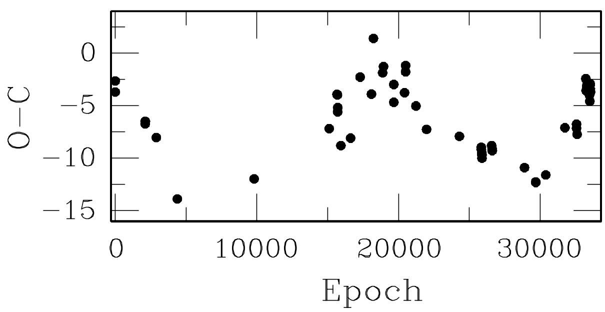

Figure 8. The $O-C$ diagram of the photoelectric photometry of XY UMa, as tabulated by Sowell et al. (2001). The scale on the vertical axis is in units of 0.001 d.

Table 4. Results of fitting each of the four models to the XY UMa data. The estimated standard deviations are given in units of days. The last two columns are the Akaike and Bayes model probabilities [see equation (10)].

Model	σ_e	σ_η	σ_ξ	p_A	p_B
		All data			
M1	4.4×10^{-3}	0	0	0.00	0.00
M2	6.6×10^{-4}	1.5×10^{-4}	0	0.75	0.88
M3	1.7×10^{-3}	0	4.5×10^{-8}	0.00	0.00
M4	6.6×10^{-4}	1.5×10^{-4}	4.0×10^{-15}	0.25	0.12

M2. The acfs of the pseudo-residuals of M2 and M4 are consistent with white noise, while the Q-statistic for model M3 is marginally significant.

Sowell et al. (2001) concluded that there are no period changes in UV Psc and YY Gem, whereas the periods of both CG Cyg and XY UMa were subject to changes. The results in Tables 1–4 are more detailed, in that models 2 and 4 are also entertained. Indeed, in this paper it is found that M2 (intrinsic period jitter) is a much more likely explanation of the $O-C$ diagram of XY UMa than M3 (a systematic period change). The UV Psc and CG Cyg data also show the presence of random period fluctuations, superimposed on a systematic period change in the case of CG Cyg. In the case of YY Gem it is not possible to choose unambiguously between M2 and M3.

6 CONCLUSIONS

A brief summary of the steps in an analysis follows.

(1) Determine the mean period as in (4). Using this value of $\overline{P}$, find the cumulative cycle numbers N_j and hence the $Z_j = (O-C)_j$ values [as in (3)].

(2) For each of the models M1–M4, the log likelihood function is maximized with respect to its unknown parameters. The entries in the appropriate covariance matrix $\mathbf{\Sigma}$ are calculated using the formulae (A6) and (A7).

(3) The information criteria *AIC* and *BIC* follow from (9), and the corresponding probabilities from (10) [and an analogous equation for the $p_B(i)$].

(4) Calculate pseudo-residuals as in (13), where $\mathbf{L}$ is defined in (12). Plot these.

(5) The residuals can be further assessed using (15) and (16).

Equation (8) is based on the assumption that the $O-C$ values have a joint normal distribution. Provided that the measurement errors e_i are Gaussian, the assumption can be justified by referring to equation (A4) in the appendix:

$$Z_j = e_j + \left(\frac{N_j}{N} - 1\right) e_0 - \frac{N_j}{N} e_{K+1} + \sum_{k=1}^{N_j} \left(\eta_k + \sum_{i=1}^{k} \xi_i\right) - \frac{N_j}{N} \sum_{k=1}^{N} \left(\eta_k + \sum_{i=1}^{k} \xi_i\right). \quad (17)$$

For reasonably large cumulative cycle numbers N_j, the sums over terms in η_k and ξ_i will each be normally distributed, by the central limit theorem (e.g. Mood, Graybill & Boes 1974). Therefore, provided that the e_i are Gaussian, Z_j in (17) is a sum over normally distributed variates, which is again Gaussian (e.g. Mood et al. 1974).

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APPENDIX A: THE COVARIANCES OF THE $O - C$ VALUES

The jth $O - C$ value, denoted by Z_j, is

$$Z_j = t_j - (t_0 + N_j \overline{P}), \quad j = 1, 2, \cdots, K \tag{A1}$$

[see equation (3)], where the mean period is given by (4). For the model of equations (1), (2) and (5) it follows that

$$\begin{aligned} t_j &= T_j + e_j = T_0 + \sum_{k=1}^{N_j} P_k + e_j \\ &= T_0 + \sum_{k=1}^{N_j} (\eta_k + \mu_k) + e_j \\ &= T_0 + N_j \mu_0 + \sum_{k=1}^{N_j} \left(\eta_k + \sum_{i=1}^{k} \xi_i \right) + e_j . \end{aligned} \tag{A2}$$

Therefore

$$\overline{P} = \frac{1}{N} \left[e_{K+1} - e_0 + N\mu_0 + \sum_{k=1}^{N} \left(\eta_k + \sum_{i=1}^{k} \xi_i \right) \right] . \tag{A3}$$

Substituting (A2) and (A3) into (A1),

$$\begin{aligned} Z_j &= e_j - e_0 + N_j \mu_0 + \sum_{k=1}^{N_j} \left(\eta_k + \sum_{i=1}^{k} \xi_i \right) - \frac{N_j}{N} \left[e_{K+1} - e_0 + N\mu_0 + \sum_{k=1}^{N} \left(\eta_k + \sum_{i=1}^{k} \xi_i \right) \right] \\ &= e_j + \left(\frac{N_j}{N} - 1 \right) e_0 - \frac{N_j}{N} e_{K+1} + \sum_{k=1}^{N_j} \left(\eta_k + \sum_{i=1}^{k} \xi_i \right) - \frac{N_j}{N} \sum_{k=1}^{N} \left(\eta_k + \sum_{i=1}^{k} \xi_i \right) . \end{aligned} \tag{A4}$$

The expected value of all the Z_j is zero. Evaluation of the covariances is straightforward, except for those involving terms in

$$U_k = \left(\sum_{j=1}^{N_k} \sum_{i=1}^{j} \xi_i \right) = \sum_{j=1}^{N_k} (N_k - j + 1) \xi_j ,$$

and these are dealt with in detail. Assuming $k \leqslant \ell$,

$$\begin{aligned} \mathrm{cov}(U_k, U_\ell) &= \mathrm{cov} \left[\sum_{j=1}^{N_k} (N_k - j + 1) \xi_j ,\ \sum_{i=1}^{N_\ell} (N_\ell - i + 1) \xi_i \right] \\ &= \mathrm{cov} \left[\sum_{j=1}^{N_k} (N_k - j + 1) \xi_j ,\ \sum_{i=1}^{N_k} (N_\ell - i + 1) \xi_i \right] \\ &= \sigma_\xi^2 \sum_{j=1}^{N_k} (N_k - j + 1)(N_\ell - j + 1), \end{aligned}$$

which is easily shown to reduce to

$$\mathrm{cov}(U_k, U_\ell) = \frac{1}{6} N_k (N_k + 1) \sigma_\xi^2 (3N_\ell - N_k + 1) . \tag{A5}$$

It follows from (A4) and (A5) that

$$\begin{aligned} \mathrm{var}(Z_j) &= 2 \left[\left(\frac{N_j}{N} \right)^2 - \frac{N_j}{N} + 1 \right] \sigma_e^2 + N_j \left(1 - \frac{N_j}{N} \right) \sigma_\eta^2 \\ &\quad + \frac{N_j}{6} \left[(N_j + 1)(2N_j + 1) - 2 \frac{N_j}{N} (N_j + 1)(3N - N_j + 1) + \frac{N_j}{N} (N + 1)(2N + 1) \right] \sigma_\xi^2 . \end{aligned} \tag{A6}$$

Similarly,

$$\mathrm{cov}(Z_k, Z_\ell) = \left[\left(1 - \frac{N_k}{N}\right)\left(1 - \frac{N_\ell}{N}\right) + \frac{N_k N_\ell}{N^2}\right]\sigma_e^2 + N_k\left(1 - \frac{N_\ell}{N}\right)\sigma_\eta^2$$
$$+ \frac{N_k}{6}\left[(N_k+1)(3N_\ell - N_k + 1) - \frac{N_\ell}{N}(N_k+1)(3N - N_k + 1) - \frac{N_\ell}{N}(N_\ell+1)(3N - N_\ell + 1) + \frac{N_\ell}{N}(N+1)(2N+1)\right]\sigma_\xi^2 \tag{A7}$$

for $k < \ell$.

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