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MULTIPLICATION OF CROWNS

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ABSTRACT. It is known that the only finite topological spaces that are H-spaces are the discrete spaces. For a finite poset which is weakly equivalent to an H-space, a generalized multiplication may be found after suitable subdivision. In this paper we construct minimal models of the k-fold generalised multiplications of circles in the category of relational structures, including poset models. In particular, we obtain higher dimensional analogues of a certain ternary multiplication of crowns [Hardie and Witbooi, *Topology Appl.* 154 (2007), no. 10, 2073–2080].

1. Introduction

A categorical equivalence between the category of finite posets and the category of finite T_0 spaces is discussed in the paper [8] of McCord. Results from [8] imply that for a suitable topological space, in particular for a compact polyhedron, there exists a finite poset *model* in the sense of weak homotopy equivalence. For the relevant basic notions of algebraic topology we refer the reader to the textbook [7] of J. P. May. Thus for instance, *crowns* are poset models of the circle. A poset model of (a generalized version of) circle multiplication appears in [3] and is used to produce a finite model of the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$. A ternary multiplication with the relevant associativity property appears in [4]. In view of the result of Stong [9] on homotopical triviality of a finite topological space that admits a multiplication, it turns out that some subdivision on the domain side is necessary. Thus for instance in [3] the multiplication is a poset map from the product of two 8-point crowns to a 4-point crown.

For a given topological space, we may also want to find a finite model of least cardinality, i.e., *a minimal* poset model. See the paper [1] of Barmak and Minian in this regard. Modeling of topological spaces is possible in the bigger category $\mathcal{R}$ of *relational structures*, defined in the sequel. Such models have been described in [5] and in [2] for instance.

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In this article we shall consistently use N to denote the set of all non-negative integers, i.e.,

$$N = \mathbb{N} \cup \{0\}\,.$$

We show how a certain sequence of functions $\mu_k : N^k \to N$ give rise to finite poset models of k-fold circle multiplications. In Section 2 we define the functions μ_k and observe how they induce the functions $m_k : \mathbb{Z}_{4k}^k \to \mathbb{Z}_4$. In Section 4 we prove that each m_k turns out to be a poset map from a product of $(4k)$-point crowns to a 4-point crown. In this way we obtain poset models of k-fold circle multiplication. Also in Section 4, we obtain some further *relational models* of the said circle multiplication. The basic structures and models are presented in Section 3.

2. Defining the function

Notation 2.1. Recall, $N = \mathbb{N} \cup \{0\}$. For $k \in \mathbb{N}$ we define the following:

(a) For each $x \in N^k$, we write

$$x_* = \min\{x_1, x_2, x_3, \dots, x_k\} \quad \text{and} \quad x^* = \max\{x_1, x_2, x_3, \dots, x_k\}\,.$$

(b) We use the notation, for $x \in \mathbb{R}^k$, that $\|x\|$ denotes the Euclidean norm, and $\|x\|_0 = \max\{|x_1|, |x_2|, \dots, |x_k|\}$.

(c) Let $\mathbb{M}(k) = \{x \in N^k : x_1 \ge x_2 \ge x_3 \ge \cdots \ge x_k\}$.

(d) For $x \in N^k$ and $d \in N$ we can form a point $x \times d \in N^{k+1}$ in an obvious way.

(e) Note that for every $x \in N^k$, it is possible to permute the coordinates of x so as to obtain an element $\bar{x} \in \mathbb{M}(k)$.

Now we define a sequence of functions $\mu_k : N^k \to N$ inductively as follows:

Item 2.2. For $x \in N$ we let $\mu_1(x) = \min\{x, 4\}$. Given any $x \in \mathbb{N}^k$, let $y = \bar{x}$. We define λ_1, and then inductively the λ_i for each $i \in \{2, 3, 4, \dots, k\}$, and finally μ_k as follows:

$$\begin{aligned} \lambda_1(x) &= \min\{4, y_1 - y_2\}, \\ \lambda_i(x) &= \min\{4i, \lambda_{i-1}(x) + y_i - y_{i+1}\}, \\ \mu_k(x) = \lambda_k(x) &= \min\{4k, \lambda_{k-1}(x) + y_k\}. \end{aligned}$$

Immediately we note that if $x \in N^k$ and every coordinate of x is even, then $\mu_k(x)$ is even. Note also that if $x \in N^k$ and $y \in N^{k+1}$ is obtained by inserting 0 as an additional coordinate into x (any position), then $\mu_{k+1}(y) = \mu_k(x)$. Figure 1 is a diagrammatical representation of μ_2. We form it by starting with a grid representing a subset of N^2 and then we replace each point x of N^2 by the number $\mu_2(x)$. For better clarity we suppress some occurrences of 5 and 7.

The next result follows readily, and we omit the proof.

Proposition 2.3. *Let $k > 1$, consider any $x \in N^k$ and let $y = \bar{x} \in \mathbb{M}(k)$. Suppose that for each $i \in \{2, 3, 4, \dots, k\}$ we have $y_1 - y_i \le 4(i-1)$. Then $\lambda_{k-1}(x) = y_1 - y_k$.*

4		6		8	8	8	8	8	8
4		6		8	8	8	8	8	8
4		6					7	8	8
4		6	6	6	6	6		8	8
4					5	6		8	8
4	4	4	4	4		6		8	8
3	3	3	3	4		6			
2	2	2	3	4		6	6	6	6
1	1	2	3	4					
0	1	2	3	4	4	4	4	4	4

Figure 1.

Proposition 2.4. *Consider a pair of points $x, x' \in N^k$. Suppose that for some $j \in \{1, 2, \ldots, k\}$ we have $x_j = x'_j + 1 > 4k$ and $x_i = x'_i \leq x'_j$ for each $i \neq j$. Then $\mu_k(x) = \mu_k(x')$.*

Proof. The case $k = 1$ is clearly true. We assume now that $k > 1$. Consider such x, $x' \in N^k$ and let $y = \bar{x}$ and let $y' = \bar{x}'$. First note that if $y_1 - y_2 > 4$, then $y'_1 - y'_2 \geq 4$. Therefore, $\lambda_1(x) = 4 = \lambda_1(x')$ and then it easily follows that $\mu(x) = \mu(x')$. Thus henceforth we assume that $y_1 - y_2 \leq 4$. We consider two cases.

Case 1: Suppose that for every $i \in \{1, 2, 3, \ldots, k-1\}$ we have that $y_1 - y_i \leq 4(i-1)$. Then by Proposition 2.3,

$$\lambda_{k-1}(x) = y_1 - y_k \quad \text{and} \quad \lambda_{k-1}(x') = y'_1 - y_k.$$

Therefore

$$\mu_k(x') = \min\{4k, \lambda_{k-1}(x') + y_k\} = \min\{4k, y_k\} = 4k.$$

Similarly, $\mu_k(x) = 4k$.

Case 2: Suppose that for some $t \in \{1, 2, 3, \ldots, k-1\}$ we have that $y_1 - y_t > 4(t-1)$, and choose i to be the smallest among such t. Then $\lambda_i(x) = 4(i-1)$ and $\lambda_i(x') = 4(i-1)$. Thus it turns out that $\mu_k(x) = \mu_k(x')$. □

We set the notation for the pivotal result, Theorem 2.5 below. Let $F_k = \{x \in N^k : x_i \leq 4k$, for each $i = 1, 2, \ldots, k\}$. From F_k we can form a new object G_k by identifying certain pairs of points. These pairs are all those of the form $\{x, y\}$ such that for some index j we have $x_j = 0$ while $y_j = 4k$, and whenever $i \neq j$ then $x_i = y_i$. The set G_k can be regarded as $\mathbb{Z}_{4k}^k$. Here $\mathbb{Z}_n$ denotes, of course, the integers modulo n.

Theorem 2.5. *The function μ_k induces a (well-defined) function*

$$m_k : \mathbb{Z}_{4k}^k \to \mathbb{Z}_4.$$

Proof. We need to prove that whenever a pair of points x and y are identified towards formation of G_k, then

$$\mu_k(x) \equiv \mu_k(y) \mod 4.$$

Since $\mu_1(0) = 0$ and $\mu_1(4) = 4$, it follows that the case $k = 1$ of the proposition is true. For higher dimensions it suffices to show that the statement below is true:

If $x \in N^k$ and $l \in N$ with $l \geq 4(k+1) \geq x^*$, then

$$\mu_{k+1}(x \times l) = \mu_{k+1}(x \times 0) + 4.$$

Now consider any $x \in N^k$ and suppose that $l \geq 4(k+1) \geq x^*$. By repeated application of Proposition 2.4 we observe that for any $x \in N^k$,

$$\mu_{k+1}(x \times (4k+4)) = \mu_{k+1}(x \times (4k+5)) = \mu_{k+1}(x \times (4k+6)) = \dots.$$

Thus it suffices to assume that $l = 4k + 8$, in which case $\lambda_1(x \times l) = 4$. Then for every $i \in \{1, 2, 3, \dots, k\}$ we have that

$$\lambda_{i+1}(x \times l) = 4 + \lambda_i(x \times 0),$$

and in particular then,

$$\mu_{k+1}(x \times l) = \lambda_{k+1}(x \times l) = 4 + \lambda_k(x \times 0) = 4 + \mu_{k+1}(x \times 0).$$

□

3. Relational structures

Definition 3.1. (See Larose and Tardif [6]). A *reflexive binary relational structure* $\mathbf{X} = (X, \theta)$ is a set X equipped with a reflexive binary relation

$$\theta \subseteq X \times X.$$

The category of which the objects are the binary reflexive relational structures and the morphisms are the functions $f : (X, \theta_X) \to (Y, \theta_Y)$ satisfying the condition: $(x_1, x_2) \in \theta_X \Rightarrow (fx_1, fx_2) \in \theta_Y$, will be denoted by $\mathcal{R}$. For short we shall refer to $\mathcal{R}$ as the category of *relational structures*.

The category **Poset** of partially ordered sets (or simply *posets*) is a full subcategory of $\mathcal{R}$, and provides an important connection between $\mathcal{R}$ and the category **Top** of topological spaces and continuous functions.

For any finite poset X and any $x \in X$, let $U_x = \{y \in X : (y, x) \in \theta_X\}$. Let $T(X)$ be the topology generated by the subbase $\mathcal{U} = \{U_x : x \in X\}$. If V is any finite topological space that satisfies the T_0 separation axiom, then we can form an $\mathcal{R}$-object $R(V)$. We define a relation θ_V on V as follows: $(x, y) \in \theta_V$ if and only if y belongs to the closure of the set $\{x\}$. In this way we obtain functors T and R which constitute an equivalence between the categories of finite posets and the finite T_0-spaces. Therefore we can refer to a finite poset as a topological space.

Definition 3.2. (See [6]). The *product* of two relational structures (X, θ_X) and (Y, θ_Y) is defined to be the relational structure $(X \times Y, \theta_{X\times Y})$, where $\theta_{X\times Y}$ is the relation on $X \times Y$ described as follows:

For x, $x_1 \in X$ and $y, y_1 \in Y$,

$$((x, y), (x_1, y_1)) \in \theta_{X\times Y} \text{ if and only if } (x, x_1) \in \theta_X \text{ and } (y, y_1) \in \theta_Y.$$

Example 3.3. We consider three different relational structures on $\mathbb{Z}$:

(a) Let $X_0 = (\mathbb{Z}, \theta_0)$ have the poset relation, or partial order, given by

$$(k, l) \in \theta_0 \Leftrightarrow l = k \text{ or } |k - l| = 1 \text{ and } k \text{ is even.}$$

(b) Let $X_1 = (\mathbb{Z}, \theta_1)$ be the (antisymmetric) relational structure on the set $\mathbb{Z}$, for which we consider $(n, m) \in \theta_1$ if and only if $m - n \in \{0, 1\}$.

(c) Let $X_2 = (\mathbb{Z}, \theta_2)$ be the (symmetric) relational structure on the set $\mathbb{Z}$, for which we consider $(n, m) \in \theta_2$ if and only if $|m - n| \leq 1$.

Notation 3.4. (a) For any of the relational structures θ_t defined in Example 3.3, there is an induced structure on subsets and products of subsets. Thus we have induced relational structures on N^k. These induced relational structures will be denoted by the same symbol.

(b) In the rest of the paper, for $x, y \in N^k$ the statement $(x, y) \in (N^k, \theta_0)$ will be denoted by $x \to y$. Likewise the relations of θ_1 and θ_2 will be denoted by $\rightharpoonup$ and $\rightleftharpoons$ respectively. By $\leq$ we shall denote the usual order on $\mathbb{R}$.

(c) For $k \in \mathbb{N}$ and $t = 0, 1, 2$, let $X_t(4k)$ denote the sub-relational structure determined by the subset $\{0, 1, 2, \ldots, 4k\}$ of X_t and let $C_t(4k)$ be the set obtained from $X_t(4k)$ by identifying the points 0 and $4k$, together with the structure θ_t. The poset $C_0(4k)$ is called the $4k$-point *crown*.

We record the following observations for easy reference.

Remark 3.5. Consider any $x, y \in N^k$. Then,

$x \rightharpoonup y$ if and only if for each i we have $0 \leq y_i - x_i \leq 1$,

$x \rightleftharpoons y$ if and only if for each i we have $|y_i - x_i| \leq 1$,

$x \to y$ if and only if ($\|x - y\|_0 \leq 1$ and $x_i = y_i$ whenever x_i is odd).

Definition 3.6. (See [6]). Let us fix an $\mathcal{R}$-morphism $g : X \to Y$. A finite subset $\{x_1, x_2, \ldots, x_k\}$ of X is said to be a *chain* in X if for every $i, j \in \{1, 2, \ldots, k\}$ with $i < j$, we have $(x_i, x_j) \in \theta_X$. Let X' be the collection of all finite chains in X. Then X' is a poset (under subset inclusion), called the *barycentric subdivision* of X. For a chain C in X, the subset $g(C)$ is a chain in Y, and so there is a poset morphism $g' : X' \to Y'$ making barycentric subdivision to be a functor from $\mathcal{R}$ to **Poset**.

Definition 3.7. Recall that a morphism $g : X \to Y$ in **Top** is said to be a *weak equivalence* or more precisely a *weak homotopy equivalence* if the induced morphism $g_* : \pi_0(X) \to \pi_0(Y)$ is a bijection and $g_* : \pi_i(X, x) \to \pi_i(Y, g(x))$ is an isomorphism of groups for each $i \in \mathbb{N}$ and each $x \in X$.

Two morphisms $g_1 : X_1 \to Y_1$ and $g_2 : X_2 \to Y_2$ in **Top** are said to be *weakly homotopy equivalent* if there is a homotopy commutative diagram such as given below, in which the horizontal arrows are weak homotopy equivalences.

$$\begin{array}{ccccc} X_1 & \longleftarrow & X_0 & \longrightarrow & X_2 \\ {\scriptstyle g_1}\downarrow & & {\scriptstyle g_0}\downarrow & & {\scriptstyle g_2}\downarrow \\ Y_1 & \longleftarrow & Y_0 & \longrightarrow & Y_2 \end{array}$$

Definition 3.8. An $\mathcal{R}$-morphism $g : X \to Y$ is an $\mathcal{R}$-*model* for the **Top**-morphism $f : A \to B$ if the barycentric subdivision $g' : X' \to Y'$ (which can be regarded as a **Top**-morphism) is weakly homotopy equivalent to f.

4. The main theorem

In this section we use the notation $\overline{u}$ to denote the element of N^k for which every coordinate is 1. Also, when unambiguous, we write μ_k as μ.

Proposition 4.1. *Suppose that $x, y \in N^k$ and for some j we have $y_j = x_j + 1$ while $y_i = x_i$ for $i \neq j$. Then $\mu(x) \leq \mu(y) \leq \mu(x) + 1$.*

Proof. Without loss of generality we can assume that $k \geq 2$ and $x, y \in \mathbb{M}(k)$. It is just interesting to note that $\lambda_{j-1}(y) \leq \lambda_{j-1}(x) \leq \lambda_{j-1}(y) + 1$. Nevertheless, if $j = k$ it follows immediately that $\mu_k(x) \leq \mu_k(y) \leq \mu_k(x) + 1$. If $j \leq k$, then $\lambda_j(x) \leq \lambda_j(y) \leq \lambda_j(x) + 1$ and from this eventually it turns out again that $\mu_k(x) \leq \mu_k(y) \leq \mu_k(x) + 1$. □

Theorem 4.2. *Consider any $t = 0, 1, 2$ and $k \in \mathbb{N}$. Given the relational structure θ_t on N, the function μ_k is an $\mathcal{R}$-morphism and induces an $\mathcal{R}$-morphism $m : C_t(4k)^k \to C_t(4)$.*

Proof. We first prove the case $t = 1$. It suffices to show that if $x, y \in N^k$ and $x \rightharpoonup y$, then

$$\mu(x) \leq \mu(y) \leq \mu(x) + 1.$$

Clearly, $\mu(x) \leq \mu(x + \overline{u}) \leq \mu(x) + 1$. We can find a sequence as below, with $y = x^{(r)}$ for some r,

$$x = x^{(1)} \rightharpoonup x^{(2)} \rightharpoonup x^{(3)} \rightharpoonup \cdots \rightharpoonup x^{(r)} = x + \overline{u},$$

such that in view of Proposition 4.1 we have

$$\mu(x^{(r)}) \leq \mu(x^{(r+1)}) \leq \mu(x^{(r)}) + 1.$$

Then we have $\mu(x) \leq \mu(y) \leq \mu(x + \overline{u}) \leq \mu(x) + 1$. This settles the $(t = 1)$-case.

Now we prove the $(t = 0)$-case. Consider any $x, y \in N^k$ with $x \to y$. From the $(t = 1)$-case it follows that $|\mu(y) - \mu(x)| \leq 1$. Now we need to prove that it is impossible to have:

$$\mu(x) \text{ being odd while } \mu(y) \text{ is even.} \tag{1}$$

Let y° be the point with

$$y_i^\circ = x_i \text{ if } y_i > x_i, \quad \text{and} \quad y_i^\circ = y_i \text{ otherwise.}$$

There exist $x_{\rm ev}, y^{\rm ev} \in N^k$ such that $x_{\rm ev}$ and $y^{\rm ev}$ have no odd coordinates and with the following relations:

$$x_{\rm ev} \rightharpoonup x \rightharpoonup x_{\rm ev} + \overline{u}, \quad \text{and} \quad y \rightharpoonup y^\circ \rightharpoonup y^{\rm ev}.$$

Furthermore, $\mu(x_{\rm ev})$ and $\mu(y^{\rm ev})$ are even integers. Now using the latter fact together with the aforegoing relations, it can be shown that the situation (1) above can never arise. This completes the proof of the $(t = 0)$-case.

Finally, the proof of the $(t = 2)$-case leans on the $(t = 1)$-case, with arguments similar to (but simpler than) the proof of the $(t = 0)$-case, and we omit the detail. □

The restriction of the map m to any of the axes of $C_t^k(4k)$, for any of the relational structures θ_t, yields a model of a degree 1 selfmap of the circle, as is the case for k-fold circle multiplication. Also the diagonal $D = \{(z, z, \dots, z) \in \mathbb{S}^k\}$ is homeomorphic to a circle and the k-fold circle multiplication, when restricted to D, yields a circle map of degree k. In the discrete case, for each $t = 0, 1, 2$, the diagonal D_t of $C_t^k(4k)$ is isomorphic in $\mathcal{R}$ to the crown $C_t(4k)$. Furthermore, the morphism $m|_{D_t}$ is a model of a circle map of degree k. Also note that any sub-relational structure of $C_t(4)$ having fewer than 4 points, will fail to be a model of the circle.

Now let us focus on m in the poset case (and note that the same argument works for the other two relational structures covered in Theorem 4.2). Given that the codomain of the multiplication is a crown with 4 points, and requiring the domain to be a k-fold cartesian power $C_t^k(l)$ with the induced map from the diagonal of $C_0^k(l)$ to $C_0(4)$ to be a degree k circle map, we can ask how small we can choose l. Simple arithmetic shows that l cannot be chosen any smaller than $4k$. Our map m does in fact realize a multiplication in this minimal case $l = 4k$.

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