Taylor & Francis
Taylor & Francis Group

Sequential Analysis

Design Methods and Applications

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/lsqa20

One-sided maximal inequalities for a randomly stopped Bessel process

Cloud Makasu

To cite this article: Cloud Makasu (2023) One-sided maximal inequalities for a randomly stopped Bessel process, Sequential Analysis, 42:2, 182-188, DOI: 10.1080/07474946.2023.2193593

To link to this article: https://doi.org/10.1080/07474946.2023.2193593

Published online: 23 May 2023.

SEQUENTIAL ANALYSIS
2023, VOL. 42, NO. 2, 182–188
https://doi.org/10.1080/07474946.2023.2193593

One-sided maximal inequalities for a randomly stopped Bessel process

Cloud Makasu

Department of Maths and Applied Maths, University of the Western Cape, Bellville, Cape Town, South Africa

ABSTRACT
We prove a one-sided maximal inequality for a randomly stopped Bessel process of dimension $1 \le \alpha < 2$. For the special case when $\alpha = 1$, we obtain a sharp Burkholder-Gundy inequality for Brownian motion as a consequence. An application of the present results is also given.

ARTICLE HISTORY
Received 28 November 2022
Revised 17 February 2023
Accepted 9 March 2023

KEYWORDS
Bessel processes; Burkholder-Gundy inequalities; optimal stopping problem

1. INTRODUCTION

Burkholder-Gundy inequalities (Burkholder 1977) for Bessel processes are established in Dante DeBlassie (1987) and Graversen and Peskir (1998) without exact constants; see also Rosenkrantz and Sawyer (1977) and the related papers. The present paper concerns a class of maximal inequalities for Bessel processes of dimension $1 \le \alpha < 2$ which improves those proved in Makasu (2023). For the special case when $\alpha = 1$, we obtain a one-sided sharp Burkholder-Gundy inequality for Brownian motion as a consequence. This result extends a proposition proved in Abundo (see Abundo 2017, Proposition 3.1) and also sharpens the Burkholder-Gundy inequality established by Schachermayer and Stebegg (2018) in the Brownian motion case. Our method of proof is entirely different from those in Abundo (2017) and Schachermayer and Stebegg (2018) and is essentially a modification of the proof in Makasu (2023). We stress that the proof is mainly based on a variant of an optimal stopping problem of the running maximum process for a Bessel process in Dubins, Shepp, and Shiryaev (1994).

For convenience, we shall use similar notation as in Makasu (2023). Let $X = (X_t)_{t \ge 0}$ be a Bessel process (see Revuz and Yor 1991; Shiga and Watanabe 1973; Yasue 2004, for instance) of dimension $1 \le \alpha < 2$, starting at $x \ge 0$, which is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathbf{P})$. Suppose that $X = (X_t)_{t \ge 0}$ is given by

$$dX_t = \frac{\alpha - 1}{2X_t}\, dt + dB_t; \quad X(0) = x, \tag{1.1}$$

CONTACT Cloud Makasu cmakasu@uwc.ac.za Department of Maths and Applied Maths, University of the Western Cape, Private Bag X17, Bellville 7535, Cape Town, South Africa.

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion under $\mathbf{P}$. Existence results for strong and weak solutions of Eq. (1.1) are given in Cherny (2000).

Our main objective in this paper is to establish some lower estimates for $\mathbf{E}_x\left[\sup_{0\leq t\leq \tau} X_t\right]$ either in terms of $\mathbf{E}_x[\tau^{1/2}]$ or $\mathbf{E}_x[\tau + x^2]^{1/2}$ for any stopping time τ of X starting at $x \geq 0$. The motivation arises from various problems in sequential analysis. For example, in the case when estimating an expected reward $\mathbf{E}_x[\frac{S_\tau^m}{(1+\tau)^q}]$ for an optimal stopping problem, where $m > 1$ and $q > 0$ are fixed and $S_\tau = \sup_{0\leq t\leq \tau} X_t$.

2. MAIN RESULTS

Our main result is the following theorem. This result improves Theorem 1.1 proved in Makasu (2023).

Theorem 2.1. *Assume that $X = (X_t)_{t\geq 0}$ is a Bessel process given by (1.1) of dimension $1 \leq \alpha < 2$ fixed, and starting at $x \geq 0$. Then,*

$$\mathbf{E}_x\left[\sup_{0\leq t\leq \tau} X_t\right] \geq \sqrt{\alpha\left(\frac{\pi}{2}\right)^{\alpha}}\, \mathbf{E}_x[\tau^{1/2}] + \frac{\alpha^2\pi^{2\alpha}}{2^{2\alpha} + \alpha^2\pi^{2\alpha}}\left(1 + \frac{\pi^\alpha}{2^\alpha} - \frac{\pi^\alpha}{2^\alpha}(\alpha+1)(\alpha-1)^{\frac{\alpha}{2-\alpha}}\right)x \tag{2.1}$$

for any stopping time τ of X. Equality holds in (2.1) if and only if $X_t = |B_t|$ when $\alpha = 1$ and for any $T > 0$ there is a stopping time τ with $\mathbf{E}[\tau^{1/2}] = \sqrt{T}$.

Proof. The first part of the proof is entirely analogous to the proof of Theorem 1.1 in Makasu (2023). Hence, we shall sketch the details. Let X be given by (1.1) and let

$$Y_t = \left(\sup_{0\leq r\leq t} X_r\right) \vee y \tag{2.2}$$

for all $0 \leq x \leq y$. Define

$$u(x, y) := \inf_\tau \mathbf{E}_{x,y}\left[Y_\tau - c\int_0^\tau \frac{1}{Y_s}\, ds\right], \tag{2.3}$$

where the infimum is taken over all stopping times τ for X such that the integral in (2.3) has finite expectation, $\mathbf{E}_{x,y}$ denotes the expectation with respect to the probability law $\mathbf{P}_{x,y} := \mathbf{P}$ of the process (X, Y) starting at (x, y) with $0 \leq x \leq y$, and $c > 0$ is some constant. For $1 < \alpha < 2$, we shall show below that the positive constant c is such that

$$\frac{\alpha}{1 + (\alpha-1)^{\frac{2}{2-\alpha}}} < c < \alpha\left(\frac{\pi}{2}\right)^{\alpha} \tag{2.4}$$

which strengthens the upper estimate in the proof in Makasu (2023).

Now arguing similarly as in Dubins, Shepp, and Shiryaev (1994) and Makasu (2023), see also Peskir (1998), we have the optimal stopping time

$$\tau^* = \inf\{t > 0 | X_t \leq \theta Y_t\} \tag{2.5}$$

for the optimal stopping problem (2.3) and $0 < \theta < 1$ is the maximal root of the equation

$$f(\theta) := \frac{2}{2-\alpha}\left(1-\frac{1}{\alpha}\right)\theta^{\alpha} - \frac{1}{2-\alpha}\theta^{2} - \frac{1}{c} + \frac{1}{\alpha} = 0. \tag{2.6}$$

Then, we can easily show that there exists a maximal root θ such that

$$\theta > (\alpha-1)^{\frac{1}{2-\alpha}} \tag{2.7}$$

if and only if c is a positive constant satisfying

$$c > \frac{\alpha}{1+(\alpha-1)^{\frac{2}{2-\alpha}}} \tag{2.8}$$

for $1 < \alpha < 2$. This follows immediately by assuming in (2.6) that $f(0) < 0$ and applying the mean value theorem. Its clear that $f(1) < 0$. Assuming that $\alpha = 1$, then its obvious that $\theta = \sqrt{1-\frac{1}{c}}$ is the maximal root of Eq. (2.6) for $c > 1$. Hence, $f(0) < 0$ does not hold in this case. Then, we impose the restriction $f(0) > 0$ whenever $\alpha = 1$.

Suppose now that there exists a positive continuous function $N(\alpha)$ such that

$$f(0) < 0 < \frac{1}{\alpha} - N(\alpha) \tag{2.9}$$

holds for $1 < \alpha < 2$ and $N(1) = \frac{2}{\pi}$. Let

$$N(\alpha) = \frac{1}{\alpha}\left(\frac{2}{\pi}\right)^{\alpha} \tag{2.10}$$

for $1 \le \alpha < 2$. It follows immediately that $N(\alpha)$ is a positive continuous function satisfying the conditions $N(1) = \frac{2}{\pi}$ and $\frac{1}{\alpha} - N(\alpha) > 0$ for $1 < \alpha < 2$.

Combining (9, 10) and (2.10), it follows that

$$\frac{\alpha}{1+(\alpha-1)^{\frac{2}{2-\alpha}}} < c < \frac{1}{N(\alpha)} \tag{2.11}$$

for $1 < \alpha < 2$.

Then, using (2.10) in (2.11), we have shown (2.4). The rest of the proof to show (2.1) now follows by arguing similarly as in Makasu (2023), with minor modifications, using the inequality

$$\begin{aligned} \mathbf{E}_{x,x}\left[\sup_{0\le t\le\tau} X_t\right] \ge{}& \max\left\{\sqrt{c}, \frac{2c^2}{\sqrt{c}(1+c^2)}, \frac{2c}{1+c}\right\}\mathbf{E}_{x,x}[\tau^{1/2}] \\ &+ \max\left\{\frac{1}{2}, \frac{c^2}{1+c^2}, \frac{1}{1+c}, \frac{c}{1+c}\right\}u(x,x) \end{aligned} \tag{2.12}$$

which is a consequence of (2.3), where $u(x,x) = (1+c(\frac{\theta^2}{2-\alpha} - \frac{2\theta^{\alpha}}{\alpha(2-\alpha)} + \frac{1}{\alpha}))x$ for all $x \ge 0$ as in Makasu (2023). Passing to the limit in (2.12) as $c \uparrow \alpha(\frac{\pi}{2})^{\alpha}$ and $\theta \downarrow (\alpha-1)^{\frac{1}{2-\alpha}}$ for $1 \le \alpha < 2$, then we have (2.1). We now complete the proof by proving the sharpness of the inequality (2.1). Let $\tau = T$ be deterministic and $X_t = |B_t|$ in the special case when $\alpha = 1$, where B_t is a Brownian motion starting at zero. Then, using Proposition 3.1 in Abundo (2017), we have equality in (2.1). The proof of the theorem is now complete. □

Using Theorem 2.1, we can prove the following corollary. The result is a one-sided sharp Burkholder-Gundy inequality for Brownian motion. It extends Proposition 3.1 in Abundo (2017) and also sharpens Theorem 1.2 in Schachermayer and Stebegg (2018).

Corollary 2.1. *Let $B = (B_t)_{t\geq 0}$ be a Brownian motion starting at $x \geq 0$. Then, we have*

$$\sqrt{\frac{\pi}{2}}\,\mathbf{E}_x[\tau + x^2]^{1/2} \leq \mathbf{E}_x\left[\sup_{0\leq t\leq\tau}|B_t|\right] \tag{2.13}$$

for any stopping time τ of B. Equality holds if and only if B_t starts at zero and for any $T > 0$ there is a stopping time τ with $\mathbf{E}[\tau^{1/2}] = \sqrt{T}$.

Proof. Let a and b be non-negative real numbers. It follows that

$$ab \leq \frac{1}{2}(a^2 + b^2) \tag{2.14}$$

and

$$(a+b)^{1/2} \leq a^{1/2} + b^{1/2}. \tag{2.15}$$

Now assume that $\alpha = 1$. Then, using Theorem 2.1, we have

$$\begin{aligned}
\mathbf{E}_x\left[\sup_{0\leq t\leq\tau}|B_t|\right] &\geq \sqrt{\frac{\pi}{2}}\,\mathbf{E}_x[\tau^{1/2}] + \frac{\pi^2}{4+\pi^2}\left(1+\frac{\pi}{2}\right)x \\
&= \sqrt{\frac{\pi}{2}}\,\mathbf{E}_x[\tau^{1/2}] + \frac{\pi^2}{4+\pi^2}\left(1+\left(\sqrt{\frac{\pi}{2}}\right)^2\right)(x^2)^{1/2} \\
&\geq \sqrt{\frac{\pi}{2}}\,\mathbf{E}_x[\tau^{1/2}] + \frac{2\pi^2}{4+\pi^2}\sqrt{\frac{\pi}{2}}\,(x^2)^{1/2} \\
&\geq \sqrt{\frac{\pi}{2}}\,\mathbf{E}_x[\tau + x^2]^{1/2}
\end{aligned} \tag{2.16}$$

which follows using the elementary inequalities (15, 16) and the fact that $\frac{2\pi^2}{4+\pi^2} > 1$. This proves the inequality (2.13). The proof of the sharpness of (2.13) is analogous to that in the proof of Theorem 2.1. It follows using Proposition 3.1 in Abundo (2017). This completes the proof.

Remark 2.1. For the case of a Bessel process $X = (X_t)_{t\geq 0}$ of dimension $1 \leq \alpha < 2$ starting at $x \geq 0$ and from the proof of Theorem 2.1, we simply adapt the proof of Corollary 2.1. Hence, we have

$$\sqrt{\alpha\left(\frac{\pi}{2}\right)^{\alpha}}\,\mathbf{E}_x[\tau + x^2]^{1/2} \leq \mathbf{E}_x\left[\sup_{0\leq t\leq\tau} X_t\right] \tag{2.17}$$

for any stopping time τ of X. This improves the inequality in Corollary 2.1 proved in Makasu (2023). The inequality (2.17) is also an interesting extension of Theorem 1.2 in Schachermayer and Stebegg (2018) to the case of a Bessel process starting at $x \geq 0$. It is quite natural to ask whether the proofs of the estimates in (2.1) and (2.17) extend to the case when the dimension $\alpha > 2$. This question remains open.

Remark 2.2. It is clear that the present results (2.1) and (2.17) cover the special case when $\alpha = 2$. We also note that the choice of $N(\alpha)$ in (2.10) is not unique.

Remark 2.3. Finally, we shall leave the reader to compare the result in Corollary 2.1 with the asymptotic form obtained by Masoliver (2014) in Eq. (46).

In the next section, by applying Theorem 2.1, we establish a lower estimate of an expected reward for an optimal stopping problem arising in sequential analysis. We shall assume that the reward process is a normalized maximum of a randomly stopped Bessel process. This application is motivated by a variant of the lower bound in Theorem 1.1 of Yan and Zhu (2004).

3. APPLICATION

Let $X = (X_t)_{t\geq 0}$ be a Bessel process starting at zero, define $S_\tau = \sup_{0\leq t\leq \tau} X_t$ for any stopping time τ of X. Fix $1 \leq \alpha < 2$, and let $m = 1 + 2q$, where $q > 0$. Then, using the reverse Hölder and Young inequalities with exponents $\nu < 0$ and $\frac{1}{\nu} + \frac{1}{\rho} = 1$,

$$\begin{aligned} \mathbf{E}\left[\frac{S_\tau^m}{(1+\tau)^q}\right] &\geq \left(\mathbf{E}\left[(1+\tau)^{-q\nu}\right]\right)^{1/\nu}\left(\mathbf{E}\left[S_\tau^{m\rho}\right]\right)^{1/\rho} \\ &\geq \frac{1}{\nu}\mathbf{E}\left[(1+\tau)^{-q\nu}\right] + \frac{1}{\rho}\mathbf{E}\left[S_\tau^{m\rho}\right]. \end{aligned} \tag{3.1}$$

Now choose $\frac{1}{\nu} = -2q$ and $\frac{1}{\rho} = m$. Hence, we have

$$\begin{aligned} \mathbf{E}\left[\frac{S_\tau^m}{(1+\tau)^q}\right] &\geq -2q\mathbf{E}\left[(1+\tau)^{1/2}\right] + m\mathbf{E}[S_\tau] \\ &\geq -2q - 2q\mathbf{E}\left[\tau^{1/2}\right] + m\mathbf{E}[S_\tau] \end{aligned} \tag{3.2}$$

by using (2.15).

Applying Theorem 2.1 in (3.2), it follows that

$$\mathbf{E}\left[\frac{S_\tau^m}{(1+\tau)^q}\right] \geq \left(m\sqrt{\alpha\left(\frac{\pi}{2}\right)^\alpha} - 2q\right)\mathbf{E}\left[\tau^{1/2}\right] - 2q. \tag{3.3}$$

For the case $q = m/2$, and $0 < m < \infty$, we note that a lower estimate for the expected reward $\mathbf{E}\left[\left(\sup_{0\leq t\leq\tau}\frac{X_t}{\sqrt{1+t}}\right)^m\right]$ is established in Yan and Zhu (2004) using the well-known Lenglart domination principle (Lenglart 1977). It is shown (Yan and Zhu 2004) that

$$\mathbf{E}\left[\left(\sup_{0\leq t\leq\tau}\frac{X_t}{\sqrt{1+t}}\right)^m\right] \geq \left(\frac{\sqrt{2}}{3a_m}\right)^m \mathbf{E}\left[R_\alpha(\tau)^{m/2}\right] \tag{3.4}$$

for any stopping time τ of the process X, where $a_m = (e + em)^{(1+m)/m}$ and $R_\alpha(t) = \log(1 + \alpha \log(1 + t))$ with $\alpha > 0$.

Now using the inequality $1 + z \leq e^z$ for $z \geq 0$ in (3.3), we obtain

$$\mathbf{E}\left[\frac{S_\tau^m}{(1+\tau)^q}\right] \geq \left(m\sqrt{\alpha\left(\frac{\pi}{2}\right)^\alpha} - 2q\right)\mathbf{E}\left[\sqrt{\log(1+\tau)}\right] - 2q \tag{3.5}$$

in our case.

Note that we obtain this lower bound without using the Lenglart inequality but by simply applying the results of this paper.

ACKNOWLEDGEMENTS

The valuable suggestions and comments by the handling editors, and anonymous referees are gratefully acknowledged. It is also a pleasure to thank Professor M. Abundo at the Universita di Roma Tor Vergata (Italy) for drawing the author's attention to reference (Abundo 2017) which motivated this work. Part of this work was written while on leave of absence and visiting the Department of Maths, University of Zimbabwe.

DISCLOSURE

The author has no conflicts of interest to report.

REFERENCES

Abundo, M. 2017. "The Mean of the Running Maximum of an Integrated Gauss-Markov Process and the Connection with Its First-Passage Time." *Stochastic Analysis and Applications* 35 (3): 499–510. doi:10.1080/07362994.2016.1273784

Burkholder, D. L. 1977. "Exit Times of Brownian Motion, Harmonic Majorization, and Hardy Spaces." *Advances in Mathematics* 26 (2):182–205. doi:10.1016/0001-8708(77)90029-9

Cherny, A. S. 2000. "On the Strong and Weak Solutions of Stochastic Differential Equations Governing Bessel Processes." *Stochastics and Stochastics Reports* 70 (3-4):213–9. doi:10.1080/17442500008834252

DeBlassie, R. Dante. 1987. "Stopping Times of Bessel Processes." *The Annals of Probability* 15 (3):1044–51. doi:10.1214/aop/1176992079

Dubins, L. E., L. A. Shepp, and A. N. Shiryaev. 1994. "Optimal Stopping Rules and Maximal Inequalities for Bessel Processes." *Theory of Probability & Its Applications* 38 (2):226–61. doi:10.1137/1138024

Graversen, S. E., and G. Peskir. 1998. "Maximal Inequalities for Bessel Processes." *Journal of Inequalities and Applications* 2:621735. doi:10.1155/S102558349800006X

Lenglart, E. 1977. "Relation de Domination Entre Deux Processus." *Annales de l'Institut Henri Poincaré* 13:171–9.

Makasu, C. 2023. "On the Exact Constants in One-Sided Maximal Inequalities for Bessel Processes." *Sequential Analysis* 42 (1):35–42. doi:10.1080/07474946.2022.2150778

Masoliver, J. 2014. "Extreme Values and the Level-Crossing Problem: An Application to the Feller Process." *Physical Review. E, Statistical, Nonlinear, and Soft Matter Physics* 89 (4): 042106. doi:10.1103/PhysRevE.89.042106

Peskir, G. 1998. "Optimal Stopping of the Maximum Process: The Maximality Principle." *The Annals of Probability* 26 (4):1614–40. doi:10.1214/aop/1022855875

Revuz, D., and M. Yor. 1991. *Continuous Martingales and Brownian Motion*. New York: Springer-Verlag.

Rosenkrantz, W. A., and S. Sawyer. 1977. "L^p Estimates for Stopping Times of Bessel Processes." *Zeitschrift für Wahrscheinlichkeitstheorie Und Verwandte Gebiete* 41 (2):145–51. doi:10.1007/BF00538418

Schachermayer, W., and F. Stebegg. 2018. "The Sharp Constant for the Burkholder-Davis-Gundy Inequality and Non-Smooth Pasting." *Bernoulli* 24 (4A):2499–530. doi:10.3150/17-BEJ935

Shiga, T., and S. Watanabe. 1973. "Bessel Diffusions as a One-Parameter Family of Diffusion Processes." *Zeitschrift für Wahrscheinlichkeitstheorie Und Verwandte Gebiete* 27 (1):37–46. doi:10.1007/BF00736006

Yan, L., and B. Zhu. 2004. "A Ratio Inequality for Bessel Processes." *Statistics & Probability Letters* 66 (1):35–44. doi:10.1016/j.spl.2003.10.003

Yasue, A. 2004. "Bessel-like Processes and SDE." *Journal of Mathematics of Kyoto University* 44: 799–809.