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Full Length Research Paper

Fifth order two-stage explicit Runge–Kutta–Nyström method for the direct integration of second order ordinary differential equations

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In this paper a direct integration of second-order Ordinary Differential Equations (ODEs) of the form $y''(x) = f(x, y)$, $y(a) = y_0$, $y'(a) = y'_0$, using the Explicit Runge-Kutta-Nyström method with higher derivatives is presented. Various numerical schemes are derived and tested on standard problems. The higher-order explicit Runge-Kutta-Nyström (HERKN) method given in this paper is compared with the conventional Explicit Runge Kutta (ERK) schemes. Due to the limitation of ERK schemes in handling stiff problems, the extension to higher order derivative is considered. The results obtained show an improvement on ERK schemes.

Key words: Runge–Kutta–Nyström method, HERKN method, higher-order derivatives, second order ordinary differential equations.

INTRODUCTION

Runge-Kutta-Nyström method is a powerful numerical technique for the direct integration of second order Ordinary Differential Equations (ODEs) numerically. Second order ODEs usually arise from models in celestial mechanics, science and engineering. Many of such problems cannot be easily solved analytically. In this paper we consider second ODEs of the form:

$$y''(x) = f(x, y), \; y(x_0) = y_0, \; y'(x_0) = y'_0 \quad (1)$$

Where $f(x, y)$ is smooth.

In some cases, Equation 1 is always reduced to system of two ODEs and numerical methods for first order ODEs are used to solve them. In the literature, Sharp and Fine (1992), Sommeijer (1987), Dormand et al. (1987), Papageorgiou et al. (1998), El-Mikkawy and El-Desouky (2003) and Fudziah (2009) discussed the general techniques for solving (1) directly. It was shown that these methods have a greater advantage over reducing (1) to systems of first order ODEs with substantial gain in efficiency and lower storage requirements.

In this paper, we try to improve the Runge-Kutta-Nyström (RKN) methods by the techniques of Goeken (1999) and Akanbi et al. (2005, 2008) in which they used the method of higher derivatives as a multistep in stage evaluations to increase the order of a Runge-Kutta (R-K) method. The order condition obtained in this paper is up to order five (5) as shown in Table 1, which ordinarily should not exceed 4 for a 2-stage method. This is an improvement to the work done by earlier authors. (Fatunla, 1988; Papageorgiou et al., 1998; Fudziah, 2009).

In Materials and Methods, we give the theoretical procedure for the general theory of Higher-Order Explicit Runge-Kutta-Nyström (HERKN) methods. The steps to the derivation of these new methods are presented in derivation of 2-Stage HERKN methods, while the stability

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Table 1. Order conditions for y .

n	$O(h^n)$	
2	$b_1 + b_2 = \frac{1}{2}$	(6)
3	$b_2 c_2 = \frac{1}{6}$	(7)
4	$\frac{1}{2} b_2 c_2^2 = \frac{1}{24}$	(8)
	$b_2 a_{21} = \frac{1}{24}$	(9)
5	$\frac{1}{2} b_2 c_2^3 = \frac{1}{40}$	(10)
	$b_2 c_2 a_{21} = \frac{1}{40}$	(11)
	$b_2 d_{21} = \frac{1}{120}$	(12)

of the new methods is analyzed in stability analysis of the HERKN method. The new methods were afterwards implemented on some standard problems in this study; then we give a concluding remark.

MATERIALS AND METHODS

Here, the materials and methods needed for the derivation of the Higher Order Runge Explicit Runge-Kutta-Nyström is presented.

Theory of higher order derivative method

One of the major aims of this paper is to derive a new set of numerical schemes based on higher order derivative Runge-Kutta-Nyström technique. Consider Explicit Runge-Kutta-Nyström methods which produce approximation y_{n+1} and y'_{n+1} to $y(x_{n+1})$ and $y'(x_{n+1})$ respectively:

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{j=1}^{s} b_j k_j \,, \tag{2}$$

$$y'_{n+1} = y'_n + h \sum_{j=1}^{s} b'_j k_j \,, \tag{3}$$

where;

$$\left.\begin{aligned} k_1 &= f(x,y) \\ k_j &= f\left(x_n + c_j h, y_n + c_j h y'_n + h^2 \sum_{l=1}^{s-1} a_{jl} k_l\right), \quad j = 1,2,3,\cdots,s \end{aligned}\right\} \tag{4}$$

Introducing a multistep term d_{j1} in k_j as a higher order derivatives, we re-write (4) as:

$$\left.\begin{aligned} k_1 &= f(x,y) \\ k_j &= f\left(x_n + c_j h, y_n + c_j h y'_n + h^2\left(\sum_{l=1}^{s-1} a_{jl} k_l + d_{j1} h\left(\frac{\partial}{\partial x} + y' \cdot \frac{\partial}{\partial y}\right) f(x,y)\right)\right) \end{aligned}\right\}$$

$$j = 2,3,\cdots,s \,. \tag{5}$$

The coefficients c_j , a_{ij} , b_j , d_{j1} and b'_j of the RKN method are assumed to be real and s is the number of stages of the method. It is customary to represent R-K schemes in Butcher's array. In the same vein, the RKN methods in this paper will be presented in the Butcher's tableau which is in the form:

Table 2. Order conditions for y'.

n	$O(h^n)$	
1	$b'_1 + b'_2 = 1$	(13)
2	$b'_2 c_2 = \frac{1}{2}$	(14)
3	$\frac{1}{2} b'_2 c_2^2 = \frac{1}{6}$	(15)
	$b'_2 a_{21} = \frac{1}{6}$	(16)
4	$\frac{1}{6} b'_2 c_2^3 = \frac{1}{24}$	(17)
	$b'_2 c_2 a_{21} = \frac{1}{8}$	(18)
	$b'_2 d_{21} = \frac{1}{24}$	(19)

$$\begin{array}{c|c:c} c & A & d \\ \hline & b^T & \\ \hdashline & b'^T & \end{array}$$

Here $c = [0, c_2, c_3, \cdots, c_s]$, $b^T = [b_1, b_2, \cdots, b_s]$, $b'^T = [b'_1, b'_2, \cdots, b'_s]$, $d = [0, d_{21}, d_{31}, \cdots, d_{s1}]$ and $A = [a_{ij}]$ is $s \times s$ matrix respectively, where $[a_{ij}]$ for $j \geq i$, $a_{ij} = 0$ specially for the Higher-Order Explicit Runge-Kutta-Nyström (HERKN) method.

Derivation of 2-stage HERKN methods

For a 2 stage method, we set $s = 2$, so that:

$$y_{n+1} = y_n + hy'_n + h^2 \phi_{HERKN}(x_n, y_n; h) \quad (20)$$

$$y'_{n+1} = y'_n + h\, {}_D\phi_{HERKN}(x_n, y_n; h) \quad (21)$$

With

$$\phi_{HERKN}(x_n, y_n; h) = \sum_{j=1}^{2} b_j k_j \quad (22)$$

and

$${}_D\phi_{HERKN}(x_n, y_n; h) = \sum_{j=1}^{2} b'_j k_j \quad (23)$$

Where;

$$k_1 = f(x, y) \quad (24)$$

$$k_2 = f\left(x_n + c_2 h, y_n + c_2 h y'_n + h^2 \left(a_{21} k_1 + d_{21} h \left(\frac{\partial f}{\partial x} + y' \cdot \frac{\partial f}{\partial y} \right) \right) \right) \quad (25)$$

We substitute k_1 and k_2 as its multivariate Taylor's expansion in (8), and also compared coefficients with the multivariate Taylor's series expansion of $y(x_{n+1})$ and $y'(x_{n+1})$ as an approximation to the methods (5) for y_{n+1} and y'_{n+1} respectively. Then the following algebraic equations are obtained as order conditions for the HERKN methods and are presented in the Tables 1 and 2. For solvability of the aforementioned equations, simplifying assumption was used. Five of the order conditions for y were selected such that two of the equations having the variable b_2 and c_2 only and one of the equations having variable a_{21} were selected together with the remaining two order conditions to generate a method. We combined and solved these order conditions in such a special manner to generate various methods.

For instance solving equations 6, 7, 8, 9 and 10 yields the HERKN1 method.

From 13 and 14:

$$\sum b_i' - \sum b_i' c_i = \frac{1}{2}.$$

Hence:

$$\sum b_i' - \sum b_i' c_i = \sum b_i \quad \text{(From Tables 1 and 2).}$$

Factorizing:

$$\sum b_i'(1-c_i) = \sum b_i$$

$$b_i' = \frac{b_i}{1-c_i}.$$

Hence, $b_2' = \dfrac{b_2}{1-c_2}$ is used as a simplifying condition to obtain b_2'. This follows generally from the first order condition of y and the first two order conditions of y' in the generalized order condition listed by Dormand et al. (1987). A more general proof of simplifying assumptions is discussed in Hairer and Wanner (1987).

These equations are solved by maple software to obtain the following results as presented in the Butcher's array highlighted for respective families of HERKN methods. Eight methods are generated from Tables 1 and 2 and are labeled as HERKN1 – 8:

HERKN1:

$$\begin{array}{c|c:c} 0 & & \\ \frac{1}{2} & \frac{1}{8} & \frac{1}{40} \\ \hline & \frac{1}{6} & \frac{1}{3} \\ \hdashline & \frac{1}{3} & \frac{2}{3} \end{array}$$

HERKN2:

$$\begin{array}{c|c:c} 0 & & \\ \frac{\sqrt{30}}{10} & \frac{\sqrt{30}}{40} & \frac{\sqrt{30}}{200} \\ \hline & \frac{1}{2}-\frac{\sqrt{30}}{18} & \frac{\sqrt{30}}{18} \\ \hdashline & 1+\frac{\sqrt{30}}{-10+\sqrt{30}} & -\frac{5}{9}\left(\frac{\sqrt{30}}{-10+\sqrt{30}}\right) \end{array}$$

HERKN3:

$$\begin{array}{c|c:c} 0 & & \\ -\frac{\sqrt{30}}{10} & -\frac{\sqrt{30}}{40} & -\frac{\sqrt{30}}{200} \\ \hline : & \frac{1}{2}+\frac{\sqrt{30}}{18} & -\frac{\sqrt{30}}{18} \\ \hdashline & 1+\frac{\sqrt{30}}{10+\sqrt{30}} & -\frac{5}{9}\left(\frac{\sqrt{30}}{10+\sqrt{30}}\right) \end{array}$$

HERKN4:

$$\begin{array}{c|c:c} 0 & & \\ \frac{3}{5} & \frac{9}{50} & \frac{9}{250} \\ \hline & \frac{29}{108} & \frac{25}{108} \\ \hdashline & \frac{91}{216} & \frac{125}{216} \end{array}$$

HERKN5:

$$\begin{array}{c|c:c} 0 & & \\ \frac{1}{2} & \frac{3}{20} & \frac{1}{40} \\ \hline & \frac{1}{6} & \frac{1}{3} \\ \hdashline & \frac{1}{3} & \frac{2}{3} \end{array}$$

HERKN6:

$$\begin{array}{c|c:c} 0 & & \\ \frac{\sqrt{30}}{10} & \frac{3}{20} & \frac{\sqrt{30}}{200} \\ \hline & \frac{1}{2}-\frac{\sqrt{30}}{18} & \frac{\sqrt{30}}{18} \\ \hdashline & 1+\frac{\sqrt{30}}{-10+\sqrt{30}} & -\frac{5}{9}\left(\frac{\sqrt{30}}{-10+\sqrt{30}}\right) \end{array}$$

HERKN7:

$$\begin{array}{c|c:c} 0 & & \\ \frac{\sqrt{30}}{10} & \frac{3}{20} & \frac{\sqrt{30}}{200} \\ \hline : & \frac{1}{2}-\frac{\sqrt{30}}{18} & \frac{\sqrt{301}}{18} \\ \hdashline & 1+\frac{\sqrt{30}}{10+\sqrt{30}} & -\frac{5}{9}\left(\frac{\sqrt{30}}{10+\sqrt{30}}\right) \end{array}$$

HERKN8:

$$\begin{array}{c|c:c} 0 & & \\ \frac{3}{5} & \frac{3}{20} & \frac{3}{100} \\ \hline & \frac{2}{9} & \frac{5}{18} \\ \hdashline & \frac{11}{36} & \frac{25}{36} \end{array}$$

Stability analysis of the HERKN method

Stability of a numerical method is a property that determines the manner in which the error is propagated as the numerical computation proceeds (Sharp and Fine, 1988). Hence, it would be necessary to investigate the stability properties of the newly developed method. We consider the usual test problem:

$$y'' = \alpha y \tag{26}$$

Subject to initial conditions, $y(x_0) = y_0,\ y'(x_0) = y'_0,\ x \in [x_0, b]$, where α is a real number. We shall discuss cases when $\alpha = 0$ and $\alpha = -k^2$.

For this method, that is applying (19) and (20) on (26):

$$k_1 = \alpha y_n$$

$$k_2 = \alpha\left(y_n + c_2 h y'_n + h^2\left(a_{21}\alpha y_n + h d_{21}\alpha y'_n\right)\right)$$

Then, (19) and (20) becomes:

$$y_{n+1} = y_n + hy'_n + h^2\alpha(b_1 + b_2)y_n + h^3\alpha b_2 c_2 y'_n + h^4\alpha^2 b_2 a_{21} y_n + h^5\alpha^2 b_2 d_{21} y'_n \qquad (27)$$

$$y'_{n+1} = y'_n + h\alpha(b'_1 + b'_2)y_n + h^2\alpha b'_2 c_2 y'_n + h^3\alpha^2 b'_2 a_{21} y_n + h^4\alpha^2 b_2 d_{21} y'_n \qquad (28)$$

Representing (27) and (28) in matrix form gives:

$$\begin{bmatrix} y_{n+1} \\ y'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 + h^2\alpha(b_1 + b_2) + h^4\alpha^2 b_2 a_{21} & h + h^3\alpha b_2 c_2 + h^5\alpha^2 b_2 d_{21} \\ h\alpha(b'_1 + b'_2) + h^3\alpha^2 b'_{21} a_{21} & 1 + h^2\alpha b'_2 c_2 + h^4\alpha^2 b'_{21} d_{21} \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix}, \qquad (29)$$

and substituting the order conditions in Tables 1 and 2, (29) becomes:

$$\begin{bmatrix} y_{n+1} \\ y'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 + \frac{h^2\alpha}{2} + \frac{h^4\alpha^2}{24} & h + \frac{h^3\alpha}{6} + \frac{h^5\alpha^2}{120} \\ h\alpha + \frac{h^3\alpha^2}{6} & 1 + \frac{h^2\alpha}{2} + \frac{h^4\alpha^2}{24} \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix}. \qquad (30)$$

Case ($\alpha = 0$)

We test for consistency, for $\alpha = 0$, it is easily seen that:

$$y_{n+1} = y_n + hy'_n \qquad (31a)$$

$$y'_{n+1} = y'_n \qquad (31b)$$

Hence, the solution of (31a) and (31b) can be written as:

$$y'_n = y'_0$$

$y_n = y_0 + nhy'_0$; which is the expected result.

Case ($\alpha = -k^2$)

From (30):

let $a_{11} = 1 + \frac{h^2\alpha}{2} + \frac{h^4\alpha^2}{24}$,

$$a_{12} = h + \frac{h^3\alpha}{6} + \frac{h^5\alpha^2}{120}$$

$$a_{21} = h\alpha + \frac{h^3\alpha^2}{6}$$

$$a_{11} = 1 + \frac{h^2\alpha}{2} + \frac{h^4\alpha^2}{24}$$

Such that, (30) becomes:

$$\begin{bmatrix} y_{n+1} \\ y'_{n+1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix} \qquad (32)$$

that is, $Y_{n+1}^{(r)} = AY_n^{(r)}$. The eigenvalues λ_1, λ_2 of the matrix A are the roots of the characteristics equation of matrix A. The eigenvalues:

$$\lambda_1, \lambda_2 = \frac{1}{2}\left[a_{11} + a_{22} \pm \left[(a_{11} - a_{22})^2 + 4a_{12}a_{21}\right]^{\frac{1}{2}}\right]$$

(33)

Substituting $\alpha = -k^2$ in $a_{11},\ a_{12},\ a_{21},\ a_{22}$, (8) becomes:

$$\lambda_1, \lambda_2 = \frac{1}{2}\left[2 - h^2k^2 + \frac{h^4k^4}{12} \pm \left[\frac{h^2k^2}{180}(h^2k^2 - 6)(h^4k^4 - 20h^2k^2 + 120)\right]^{\frac{1}{2}}\right]$$

The eigenvalues obtained must be such that the roots have unit modulus for the method to be $R-$Stable. Hence, we have that, $6 \le h^2k^2 \le \infty$ and this implies that the method is $R-$stable and this interval is called the interval of periodicity. The order of the methods is given as order five since the derived HERKN were compared with the Taylors series up to order five, thereby letting the coefficients of $y^{(vi)}$ in the expansion of the HERKN method not to be equal to zero.

NUMERICAL RESULTS

Problem 1

Consider the test problem, $y'' = \lambda y$, $y(0) = y'(0) = 1$, $0 \le x \le 10$. The exact solution for $\lambda = -1$ is given by; $y(x) = \cos x + \sin x$. Numerical Solution to the problem using steplenght $h = 0.1$ and $h = 0.05$ are analyzed (Tables 3 to 6) using Maximun Norm. That is, $Max\|y_n - y(x_n)\|$ (Table 7).

Table 3. Theoretical result of problem 1 for $h = 0.1$.

x	ANALYTICAL	HERKN1	HERKN2	HERKN3	HERKN4
1.0	1.3817732907	1.3860546124	1.3851021003	1.3922161994	1.3858623475
2.0	0.4931505903	0.4945744769	0.4941087603	0.4977357419	0.4944653255
3.0	-0.8488724885	-0.8700148730	-0.8656950748	-0.8979736574	-0.8684714541
4.0	-1.4104461162	-1.4515232238	-1.4427202403	-1.5092902666	-1.4482331688
5.0	-0.6752620892	-0.6981254221	-0.6928905336	-0.7331235863	-0.6959667891
6.0	0.6807547885	0.7146310902	0.7077525045	0.7597477999	0.7122735286
7.0	1.4108888531	1.4890580149	1.4722492868	1.6013974951	1.4826615731
8.0	0.8438582128	0.8970420954	0.8850546363	0.9785794559	0.8920748091
9.0	-0.4990117766	-0.5358982928	-0.5285513071	-0.5844413571	-0.5335794791
10	-1.3830926400	-1.4964212795	-1.4719387638	-1.6630218120	-1.4871049775

Table 4. Theoretical result of problem 1 for $h = 0.1$.

x	HERKN5	HERKN6	HERKN7	HERKN8
1.0	1.3861481011	1.3851509052	1.3922161994	1.3838012555
2.0	0.4948742381	0.4942660797	0.4977357419	0.4934826446
3.0	-0.8697037419	-0.8655311273	-0.8979736574	-0.8597967573
4.0	-1.4516033813	-1.4427608863	-1.5092902666	-1.4307526960
5.0	-0.6987348345	-0.6932085641	-0.7331235863	-0.6858152979
6.0	0.7139403600	0.7073908507	0.7597477999	0.6984012201
7.0	1.4890161010	1.4722253053	1.6013974951	1.4495306580
8.0	0.8979081592	0.8855039976	0.9785794559	0.8689406769
9.0	-0.5347760684	-0.5279673186	-0.5844413571	-0.5185892502
10	-1.4961436621	-1.4717920909	-1.6630218120	-1.4390368724

Table 5. Theoretical result of problem 1 for $h = 0.05$.

x	ANALYTICAL	HERKN1	HERKN2	HERKN3	HERKN4
1.0	1.3817732907	1.3839724299	1.3834953184	1.3870475191	1.3838287306
2.0	0.4931505903	0.4936682799	0.4935173920	0.4946757633	0.4936262544
3.0	-0.8488724885	-0.8597665255	-0.8575011400	-0.8743767870	-0.8589102231
4.0	-1.4104461162	-1.4309054474	-1.4265517627	-1.4591625092	-1.4292272579
5.0	-0.6752620892	-0.6860111783	-0.6836415114	-0.7015243515	-0.6850496800
6.0	0.6807547885	0.6983414388	0.6946916932	0.7220687981	0.6969847757
7.0	1.4108888531	1.4496615821	1.4413967209	1.5038049701	1.4464479563
8.0	0.8438582128	0.8691424213	0.863620687	0.9055895561	0.8668988549
9.0	-0.4990117766	-0.5185012702	-0.5144848139	-0.5447837282	-0.5170524343
10	-1.3830926400	-1.4391430501	-1.4271659741	-1.5183374614	-1.4344856394

Problem 2

We also consider a system of second order ordinary differential equations:

$$y_1'' = -4x^2 y_1 - \frac{2y_2}{\sqrt{y_1^2 + y_2^2}}, \qquad y_1\left(\sqrt{\frac{\pi}{2}}\right) = 0, \quad y_1'\left(\sqrt{\frac{\pi}{2}}\right) = -\sqrt{2\pi}$$

$$y_2'' = -4x^2 y_2 - \frac{2y_1}{\sqrt{y_1^2 + y_2^2}}, \qquad y_2\left(\sqrt{\frac{\pi}{2}}\right) = 1, \quad y_2'\left(\sqrt{\frac{\pi}{2}}\right) = 0$$

,

Table 6. Theoretical result of problem 1 for $h = 0.05$.

x	HERKN5	HERKN6	HERKN7	HERKN8
1.0	1.3839960222	1.3835076996	1.3870475191	1.3828429681
2.0	0.4937428565	0.4935566344	0.4946757633	0.4933134408
3.0	-0.8596908273	-0.8574612204	-0.8743767870	-0.8544043540
4.0	-1.4309277413	-1.4265633060	-1.4591625092	-1.4206122273
5.0	-0.6861621548	-0.6837207264	-0.7015243515	-0.6804174788
6.0	0.6981748024	0.6946040969	0.7220687981	0.6897152905
7.0	1.4496564244	1.4413937220	1.5038049701	1.4301547047
8.0	0.8693557823	0.8637322976	0.9055895561	0.8561277085
9.0	-0.5182328836	-0.5143441648	-0.5447837282	-0.5090200192
10	-1.4390839457	-1.4271346649	-1.5183374614	-1.4109225938

Table 7. Maximun Norm.

Method	h=0.1	h=0.05
	$Max\|y_n - y(x_n)\|$	$Max\|y_n - y(x_n)\|$
HERKN1	1.13 (-01)	5.61 (-02)
HERKN2	8.88 (-02)	4.41 (-02)
HERKN3	2.80 (-01)	1.35 (-01)
HERKN4	1.04 (-01)	5.14 (-02)
HERKN5	1.13 (-01)	5.60 (-02)
HERKN6	8.87 (-02)	4.40 (-02)
HERKN7	2.80 (-01)	1.35 (-01)
HERKN8	5.59 (-02)	2.78 (-02)

$$\sqrt{\frac{\pi}{2}} \le x \le 10$$

This problem was considered by Sharp and Fine (1992). The exact solution is given by:

$$y_1(x) = \cos(x^2), \qquad y_2(x) = \sin(x^2)$$

This problem is solved using the newly derived HERKN schemes with steplenghts $h = 0.01,\ 0.001,$ and 0.0001. Their table of errors is presented in Table 8. Usually, the implementation of ERK methods on higher ODEs requires that they are first reduced to system of first order. However, the new HERKN methods are well able to handle second order ODEs directly and even systems of second order ODEs without reducing them to first order. The results are presented in Table 8.

From Table 8, HERKN8 gave the best results amongst the HERKN methods. Thus, the graph of solution using HERKN8 is hereby presented.

Problem 3

The linear scale problem with slowly varying frequency was also considered using steplenght $h = 0.01$. The problem which is given as:

$$y'' = -\log_e(2+x)y, \qquad x > 0$$

$$y(0) = 0,\ y'(0) = 1, \qquad x_f = 50$$

has no closed form solution and was solved with the new methods. The result is presented for some stepnumbers as displayed in Table 9.

DISCUSSION

In Problem 1 it is observed that HERKN methods gave convergent solution to the problem with maximum error given in Table 7, unlike the usual ERK method which will require reducing the system to a system of first order differential equations before implementation. The results obtained using the best method which is the HERKN8 is

Table 8. Table of errors for problem 2.

Method	x	h=0.01		h=0.001		h=0.0001	
		y_1	y_2	y_1	y_2	y_1	y_2
HERKN1	$\sqrt{\frac{\pi}{2}}+h$	2.01E-11	5.30E-11	1.20E-07	1.12E-07	1.32E-08	1.23E-08
	10	1.09E+00	1.24E+00	7.61E-02	6.50E-02	7.32E-03	6.13E-03
HERKN2	$\sqrt{\frac{\pi}{2}}+h$	2.49E-09	3.52E-10	9.46E-08	8.84E-08	1.04E-08	9.73E-09
	10	7.86E-01	8.39E-01	5.94E-02	5.06E-02	5.77E-03	4.83E-03
HERKN3	$\sqrt{\frac{\pi}{2}}+h$	5.50E-08	6.81E-09	2.82E-07	2.64E-07	3.11E-08	2.90E-08
	10	4.59E+00	7.99E+00	1.91E-01	1.65E-01	1.73E-02	1.45E-02
HERKN4	$\sqrt{\frac{\pi}{2}}+h$	4.37E-07	4.18E-07	1.14E-07	1.07E-07	1.21E-08	1.13E-08
	10	9.84E-01	1.06E+00	6.95E-02	5.92E-02	6.71E-03	5.62E-03
HERKN5	$\sqrt{\frac{\pi}{2}}+h$	1.04E-09	2.92E-09	1.20E-07	1.12E-07	1.32E-08	1.23E-08
	10	1.06E+00	1.27E+00	7.60E-02	6.51E-02	7.32E-03	6.14E-03
HERKN6	$\sqrt{\frac{\pi}{2}}+h$	3.00E-09	1.07E-09	9.46E-08	8.85E-08	1.04E-08	9.73E-09
	10	7.72E-01	8.52E-01	5.94E-02	5.07E-02	5.77E-03	4.83E-03
HERKN7	$\sqrt{\frac{\pi}{2}}+h$	5.50E-08	6.81E-09	2.82E-07	2.64E-07	3.11E-08	2.90E-08
	10	4.59E+00	7.99E+00	1.91E-01	1.65E-01	1.73E-02	1.45E-02
HERKN8	$\sqrt{\frac{\pi}{2}}+h$	5.25E-09	6.77E-10	6.00E-08	5.60E-08	6.60E-09	6.16E-09
	10	4.49E-01	4.27E-01	3.72E-02	3.15E-02	3.65E-03	3.06E-03

Table 9. Numerical result of problem 3.

n	x	HERKN1	HERKN2	HERKN3	HERKN4
0	0.00	0.0000000000000	0.0000000000000	0.0000000000000	0.0000000000000
500	5.00	-1.0162926474094	-1.0162926474094	-1.0162926474094	-1.0162926474094
1000	10.00	0.6070376099226	0.6070376099226	0.6070376099226	0.6070376099226
1500	15.00	1.2387518215344	1.2387518215344	1.2387518215344	1.2387518215344
2000	20.00	1.1381933458202	1.1381933458202	1.1381933458202	1.1381933458202
2500	25.00	0.8544308165754	0.8544308165754	0.8544308165754	0.8544308165754
3000	30.00	0.6346869445968	0.6346869445968	0.6346869445968	0.6346869445968
3500	35.00	0.5463705407451	0.5463705407451	0.5463705407451	0.5463705407451
4000	40.00	0.5916889962126	0.5916889962126	0.5916889962126	0.5916889962126
4500	45.00	0.7505546705521	0.7505546705521	0.7505546705521	0.7505546705521
5000	50.00	0.9838889466474	0.9838889466474	0.9838889466474	0.9838889466474

shown in Figures 1 to 3. It was observed that as the step size reduces the numerical results generated conforms to the analytical result as shown in Figure 3.

Conclusion

A 2-Stage Explicit Runge-Kutta Nyström method with higher order derivatives has been derived and implemented. This method has shown that the usual practise of reduction of second order ODEs to a systems of two first order ODEs can be avoided and the problem solved directly. Also, for a reduced stage evaluation we have a method with a higher order of convergence as seen from the order conditions obtained. The paper also shows that HERKN8 is the most accurate of all the methods:

Table 9. Contd.

n	x	HERKN5	HERKN6	HERKN7	HERKN8
0	0.00	0.0000000000000	0.0000000000000	0.0000000000000	0.0000000000000
500	5.00	-1.0162926474094	-1.0162926474094	-1.0162926474094	-1.0162926474094
1000	10.00	0.6070376099226	0.6070376099226	0.6070376099226	0.6070376099226
1500	15.00	1.2387518215344	1.2387518215344	1.2387518215344	1.2387518215344
2000	20.00	1.1381933458202	1.1381933458202	1.1381933458202	1.1381933458202
2500	25.00	0.8544308165754	0.8544308165754	0.8544308165754	0.8544308165754
3000	30.00	0.6346869445968	0.6346869445968	0.6346869445968	0.6346869445968
3500	35.00	0.5463705407451	0.5463705407451	0.5463705407451	0.5463705407451
4000	40.00	0.5916889962126	0.5916889962126	0.5916889962126	0.5916889962126
4500	45.00	0.7505546705521	0.7505546705521	0.7505546705521	0.7505546705521
5000	50.00	0.9838889466474	0.9838889466474	0.9838889466474	0.9838889466474

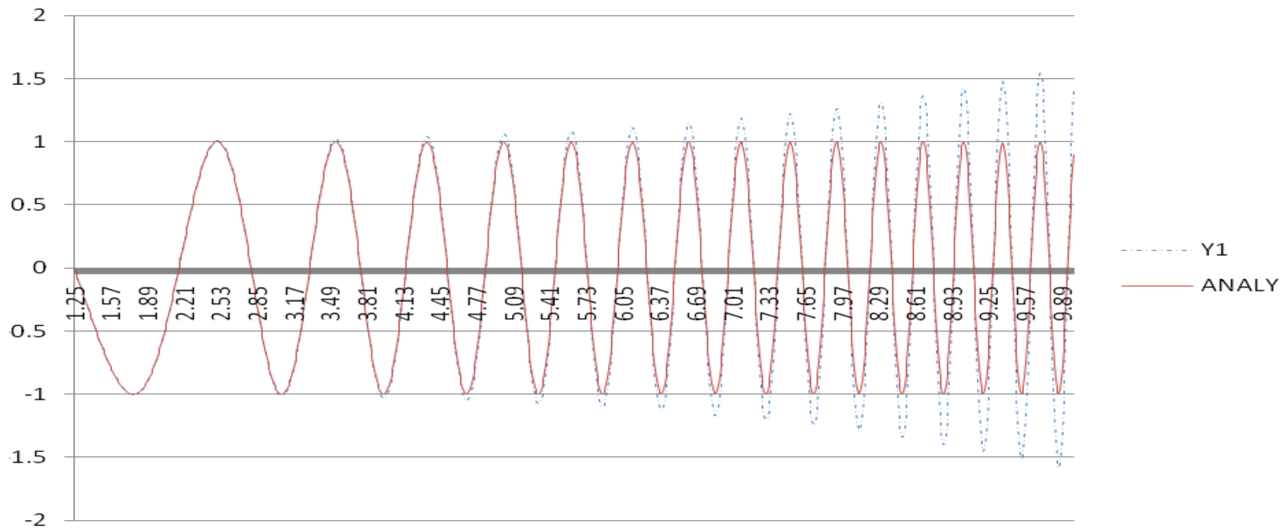

Figure 1a. Graph of Solution (*h=0.01*): HERKN8_ y_1. Notice the difference between the analytical solution and the HERKN8 method. See that the result start failing from $x = 4.05$.

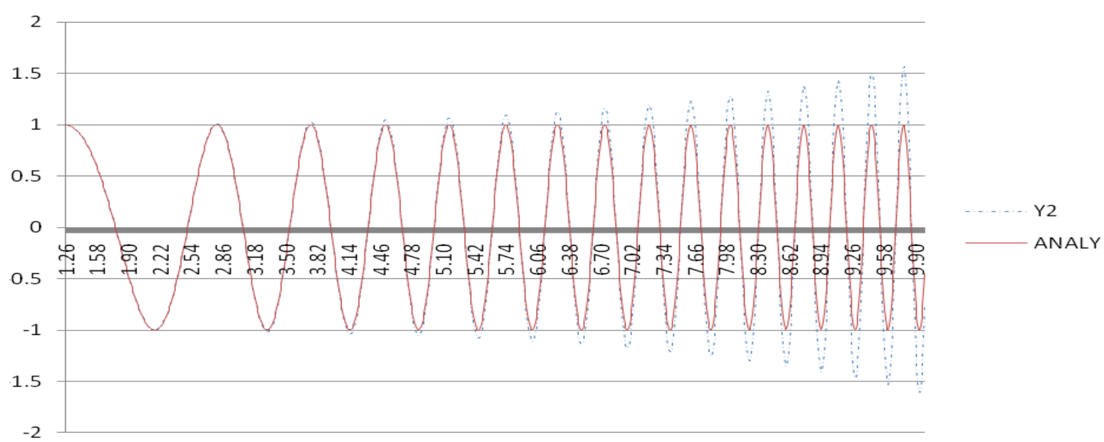

Figure 1b. Graph of Solution (*h=0.01*): HERKN8_ y_2. Notice the difference between the analytical solution and the HERKN8 method. See that the result start failing from $x = 4.26$.

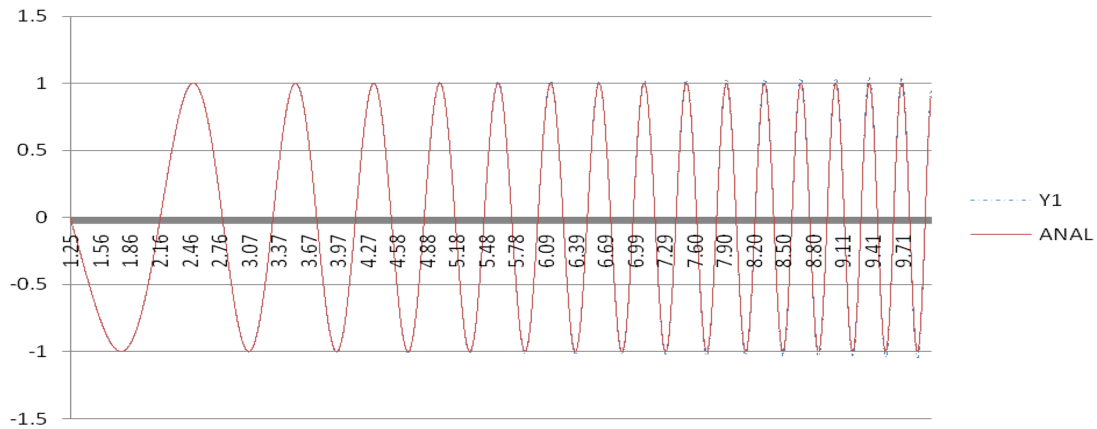

Figure 2a. Graph of Solution (h = *0.001*): HERKN8_ y_1. Notice the improvement in the difference between the analytical solution and the HERKN8 method here. See that the result start failing from $x = 8.27$.

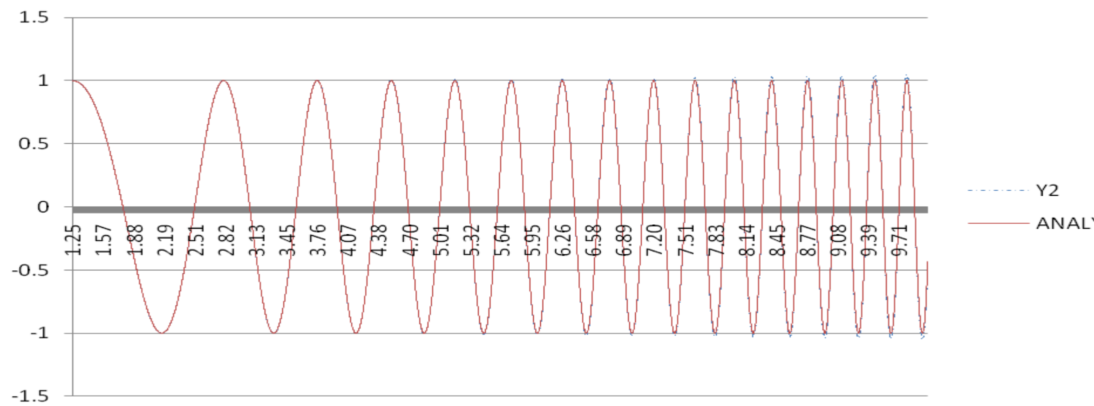

Figure 2b. Graph of Solution (h = *0.001*): HERKN8_ y_2. Notice the improvement in the difference between the analytical solution and the HERKN8 method here. See that the result start failing from $x = 8.22$.

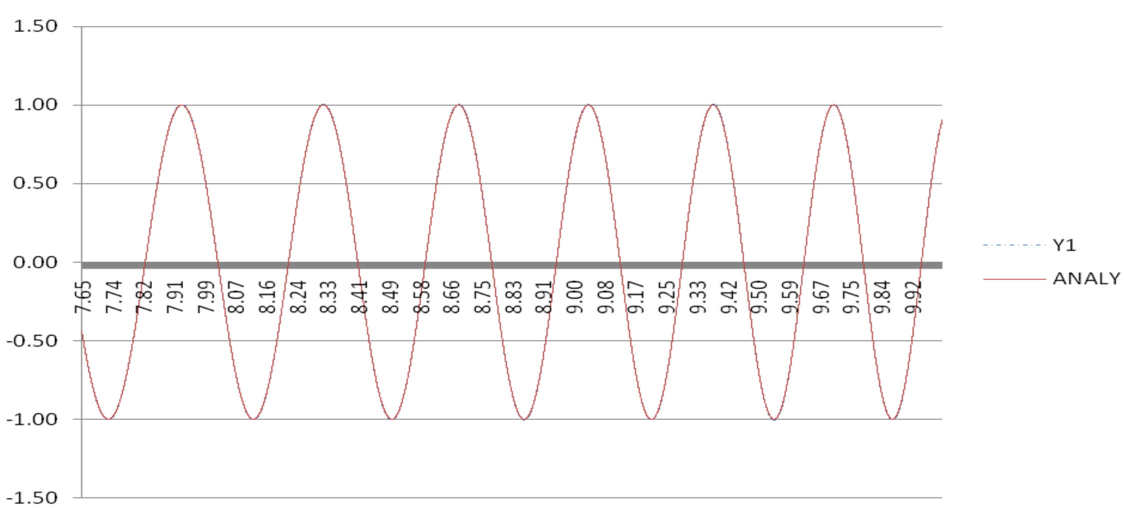

Figure 3a. Graph of solution (h = *0.0001*): HERKN8_ y_1. The graph shows a good approximation of the analytical Solution, see that the error is not visible. The graph was generated for $7.65 \le x \le 10$.

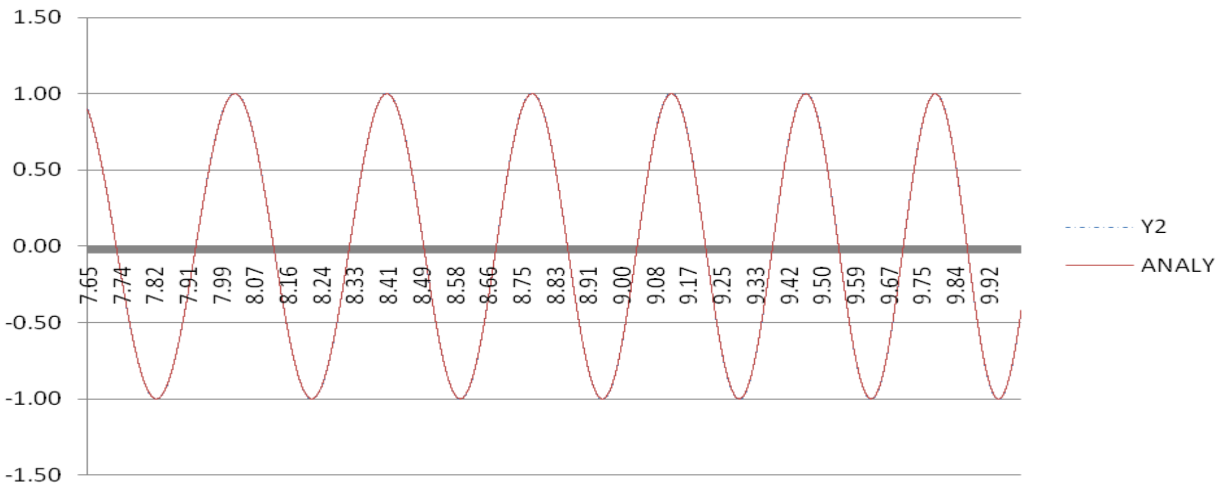

Figure 3b. Graph of Solution ($h = 0.0001$): HERKN8_ y_2 **.** The graph shows a good approximation of the analytical solution, see that the error is not visible. The graph was generated for $7.65 \le x \le 10$.

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