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The fibre of a pinch map in a model category

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Abstract

In the category of pointed topological spaces, let F be the homotopy fibre of the pinching map $X \cup CA \to X \cup CA/X$ from the mapping cone on a cofibration $A \to X$ onto the suspension of A. Gray (Proc Lond Math Soc (3) 26:497–520, 1973) proved that F is weakly homotopy equivalent to the reduced product $(X, A)_\infty$. In this paper we prove an analogue of this phenomenon in a model category, under suitable conditions including a cube axiom.

1 Introduction

In the paper [5] we have defined *reduced powers* X_n of an object X in a model category **C** and, assuming a certain cube axiom holds in **C**, established the weak equivalence,

$$X_\infty \xrightarrow{\sim} \Omega\Sigma X\,, \qquad (1.1)$$

so generalising an influential result on reduced product spaces due to James [7]. See also the generalizations in [3]. Although the argument began by constructing an analog of (X, A) (i.e. of Gray's relative version of the James construction on a cofibration $A \to X$ [4]). We were not able in [5] to recover in **C** a weak equivalence

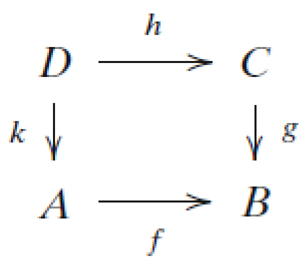
$$(X, A)_\infty \xrightarrow{\sim} F\,, \qquad (1.2)$$

where F is the homotopy fibre of the pinching map $X \cup CA \to :E A$. The situation is remedied here by showing that the desired result indeed holds under a mild additional assumption.

2 The Cube Axiom

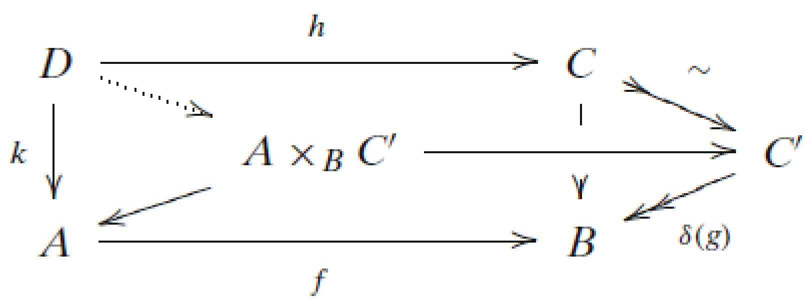
Quillen [8] described an abstract approach to homotopy theory enabling analogous theories to be defined in categories other than the category of topological spaces and continuous maps. A *model category* consists of a category **C** with all small limits and colimits together with three distinguished classes of morphisms, *we*, *cof*, *f ib*, called weak equivalences, cofibrations and fibrations, respectively. These are required to satisfy certain axioms which reflect typical properties of the classes of such maps in topology. We use the axioms as modified by Hovey, [6] and assume that **C** is *pointed*, i.e. that the initial object 0 and the terminal object $*$ are isomorphic. e. that the initial object 0 and the terminal object * are isomorphic.

A commutative diagram in **C**



(2.1)

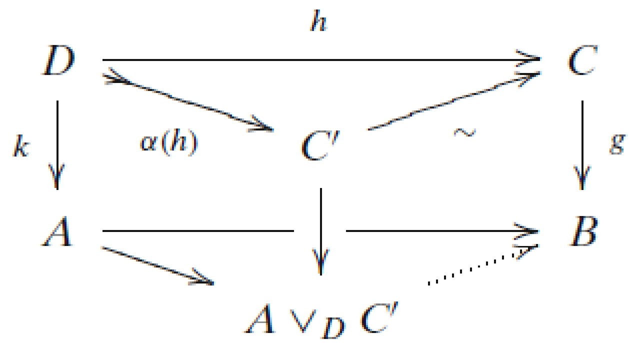
is a *homotopy pullback* if the induced map (shown dotted) in the following diagram is a weak equivalence.



(2.2)

Here it is to be understood that the square with source $A \times_B C$ is a pullback. The special case $C = *$ of Eq. 2.1 is of some significance, for then we call D the *homotopy f ibre* of f and denote it by Ff. If both $C = A = *$ then we say that D is a loop object of B and denote it by ΩB.

Dually, we define the notions of *homotopy pushout* square and *homotopy cof ibre* (i.e. *mapping cone*): specifically the square (2.1) is a homotopy pushout if the induced dotted arrow in the following diagram is a weak equivalence.

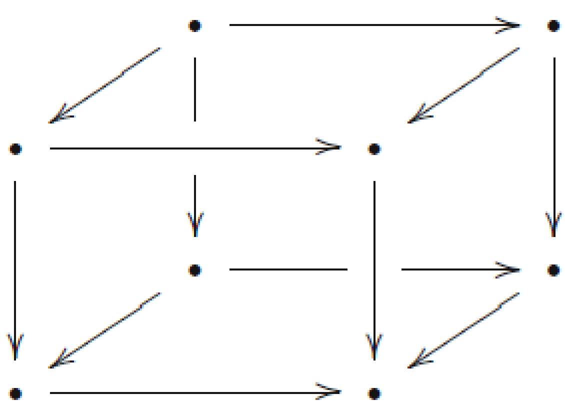


(2.3)

In the case $C = *$ of diagram (2.3), we call C a *cone* on D and $A \vee_D C$ a *mapping cone* of k. If there is a weak equivalence $X \to *$ then we say that X is *weakly contractible*. In particular each mapping cone of $1 : X \to X$ is a cone on X and is weakly contractible. A mapping cone of the final map $X \to *$ is called a *suspension* of X.

Cube axioms in abstract categories with homotopy structure feature in the book [1] of Baues. The specific forms we require are stated below, as in [5].

2.1 Cube Axioms
Suppose that we have a commutative diagram as follows.



(2.4)

1. If the top and bottom faces are homotopy pushouts and the left and rear faces are homotopy pullbacks, then the remaining two faces are homotopy pullbacks.
2. If the bottom face is a homotopy pushout and four vertical faces are homotopy pullbacks, then the top face is a homotopy pushout.

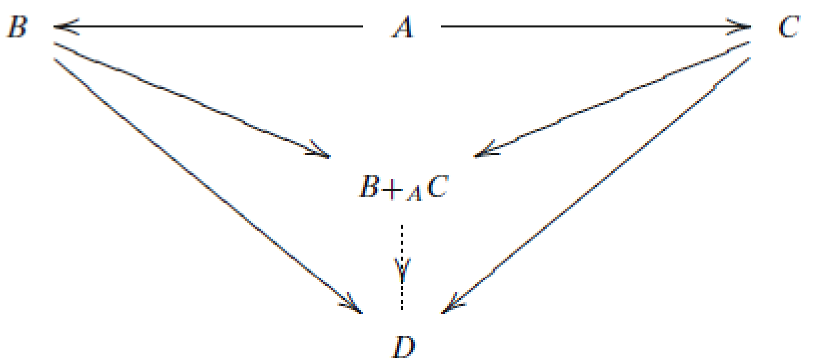
Besides the cube axioms 2.1(a) and (b) it was necessary in [5] to assume also the following.

Condition 2.1

(we) Given any object X and weak equivalence $f : A \to B$ in **C**, then the morphism $X \times f : X \times A \to X \times B$ is a weak equivalence.

(cof) Given any object X and cofibration $f : A \to B$ in **C**, then the morphism $X \times f : X \times A \to X \times B$ is a cofibration.

(cotriad) Given any commutative diagram of solid arrows as below, in which every morphism is a cofibration and the upper quadrilateral is a pushout, then the induced map $B +_A C \to$ D is a cofibration.



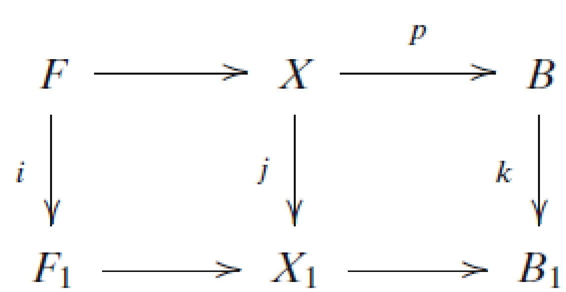
(2.5)

(lim) Given any sequence of cofibrations $A_1 \to A_2 \to A_3 \to \ldots$ in which A_n is weakly contractible for each $n > 0$, then $lim(A_n)$ is weakly contractible.

3 On Fibrations

Analogues in abstract categories of the well-known five-lemma are fairly commonly used, such as for instance in the paper [2] of Bourn and Janelidze. In order to prove the equivalence of Gray, a further such assumption is necessary. We assume throughout this section that we are working in a model category. In the formulation of the condition we use the concept of a *fibration sequence*.

Definition 3.1 The top row in diagram (3.1) is a *fibration sequence* if the composite map $F \to B$ is null-homotopic and the resulting induced map from F to the homo-map $F \to B$ is null-homotopic and the resulting topy fibre of p is a weak homotopy equivalence.



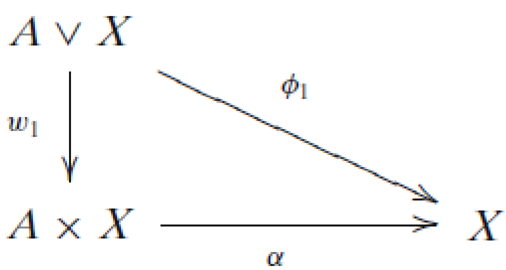
(3.1)

Condition 3.2 (On a fibration $p : X \to B$ in a model category **C**) *Suppose that we have a commutative diagram in* **C** *such as diagram* (3.1) *above, in which the horizontal sequences are f ibration sequences. If the vertical arrows i and k are weak equivalences, then the arrow j is also a weak equivalence.*

Remark 3.3 If $p : X \to B$ is a fibration in a model category and B is weakly contractible, then p satisfies Condition 3.2.

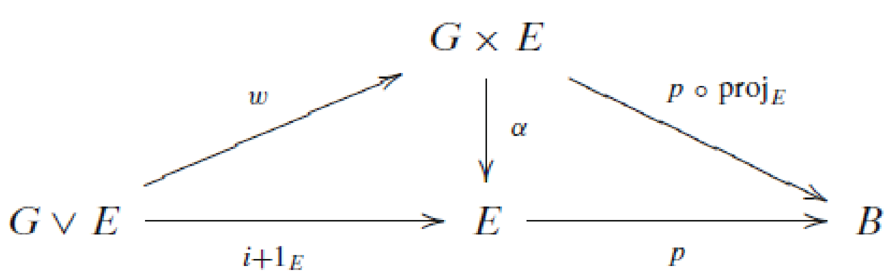
The method we follow is very similar to that of [4]. Thus we introduce the owing terminology.

Definition 3.4 An *action* of A on X is a morphism $\alpha : A \times X \to X$ for which the following square is commutative. Here φ_1 is the relevant folding map.



(3.2)

Definition 3.5 A fibration $p : E \to B$, with fibre $i : G \to E$, is a *principal f ibration* if there is an action α of G on E making the following diagram commutative.



(3.3)

Proposition 3.6 *Suppose that $p : E \to B$ is a f ibration with E contractible and such that for the fibre G, the canonical (wedge-) map $w : G \vee E \to G \times E$ is a cof ibration. Then p is a principal fibration.*

Proof Let us denote the inclusion of the fibre by $j : G \to E$. We consider the g diagram, which can be seen to be commutative.

$$
\begin{array}{ccc}
G \vee E & \xrightarrow{\quad j+1_E \quad} & E \\
{\scriptstyle w}\downarrow & & \downarrow{\scriptstyle p} \\
G \times E & \xrightarrow[\quad p\circ(j+1_E) \quad]{} & B
\end{array}
\qquad (3.4)
$$

Since w is a cofibration and E is contractible, it follows that w is a trivial cofibration. Thus by the lifting property we can fill in an arrow $\alpha : G \times E \to E$ in diagram (3.4) and the resulting diagram will still be commutative. N

The following proposition is easy to prove and we omit the proof.

Proposition 3.7 *A pull-back of a principal fibration is a principal fibration.*

4 Cones and Pinching Maps

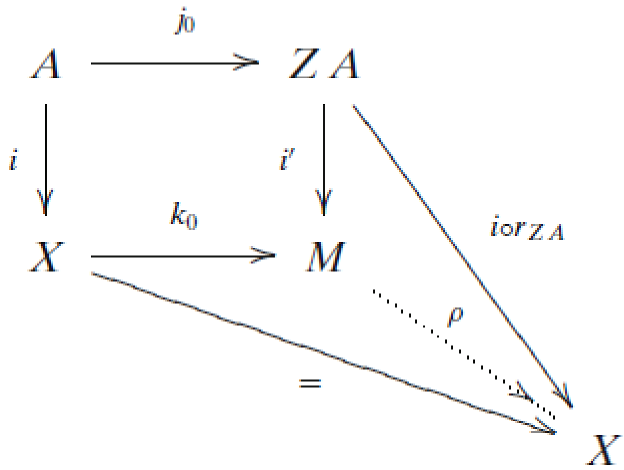
We continue working in a fixed model category. For notational convenience we describe in detail the mapping cone of a cofibration and we introduce the pinching map.

4.1 The Mapping Cone

Fix a cylinder ZA on A together with a cofibration $j : A \vee A \to ZA$ (and thus by implication two cofibrations $j_0, j_1 : A \to ZA$) and a retraction $r_{ZA} : ZA \to A$ such that $r_{ZA} \circ j_0 = 1 = r_{ZA} \circ j_1$. Then in the following pushout square,

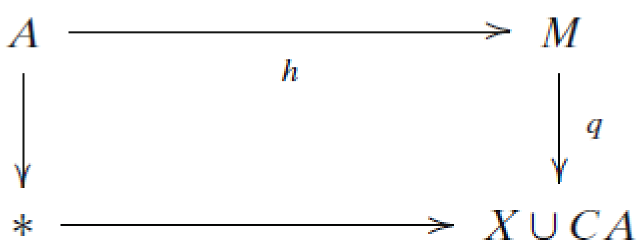
$$
\begin{array}{ccc}
A & \xrightarrow{\quad j_0 \quad} & ZA \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\
X & \xrightarrow{\quad k_0 \quad} & M
\end{array}
\qquad (4.1)
$$

M is a *mapping cylinder* of i, and k_0 is a trivial cofibration. Since Eq. 4.1 is a pushout square and in diagram (4.2) below we have $i \circ r_{ZA} \circ j_0 = i$, there exists a left inverse $\rho : M \to X$ of k_0 for which diagram (4.2) is commutative.



(4.2)

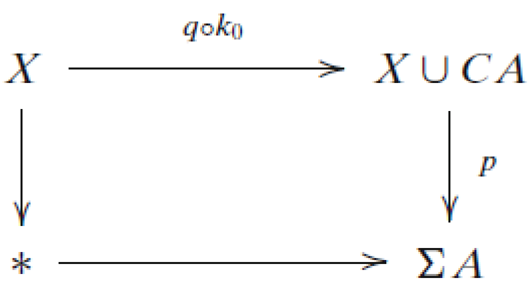
Then $h = i^{t} \circ j_1$ is a cofibration $h : A \to M$ such that the object obtained as the push-out of the cotriad (4.3) below, is a mapping cone for i and we denote it by $X \cup$.



(4.3)

4.2 The Pinching Map

Let p be the map obtained by forming the pushout of the cotriad formed by the map k_0 of diagram (4.1) together with the trivial map $X \to *$

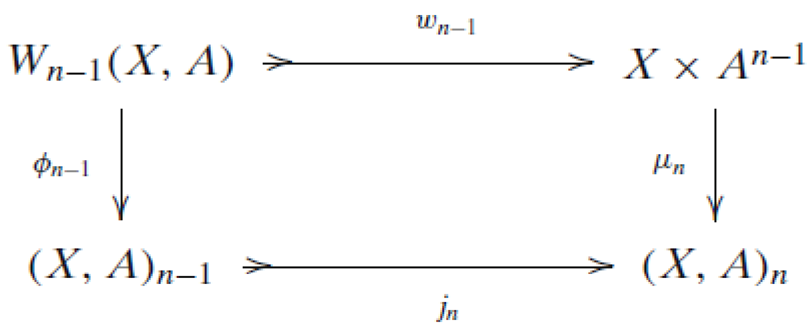


(4.4)

5 Gray's Construction and the Main Theorem

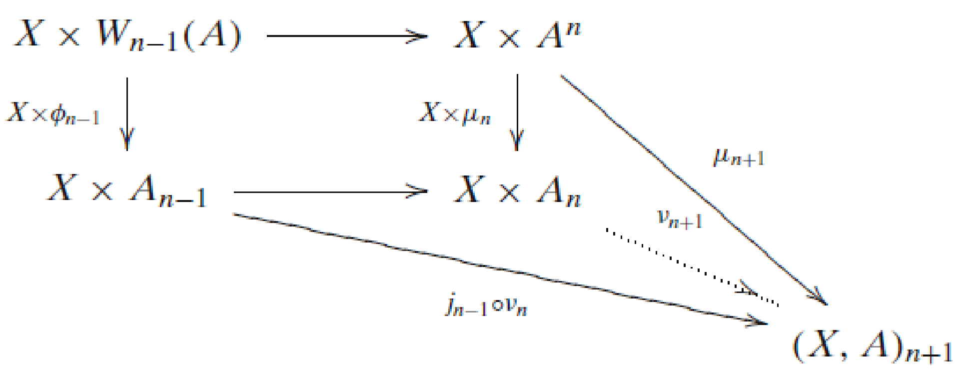
We assume throughout this section that we are working in a fixed model category **C**. In full detail, the construction in [5] of the reduced products is rather lengthy. We include here only a summary of the essential points.

For a cofibration $i : A \rightarrow X$, the following pushout square defines the objects *$(X, A)_n$*



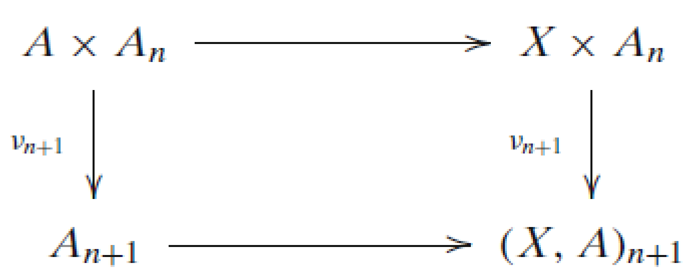
(5.1)

We note that the upper horizontal map is the analogue of the inclusion of the fat wedge and the φ_n can be regarded as folding maps. There is also a *multiplication* ν_n: $X \times A_{n-1} \rightarrow (X, A)_n$, where $\nu_2 = \mu_2$ and ν_{n+1} $(n \geq 2)$ is determined uniquely by pushout in the following.



(5.2)

An equivalent definition of the objects *$(X, A)_n$* is given by the following result.
Proposition 5.1 [5, Theorem 3.4] *The square*



(5.3)

in which the horizontal arrows are the obvious cofibrations, is a homotopy push-out.

As a final step in defining the reduced products, *$(X, A)_\infty$* is defined to be *$\lim (X, A)_n$*.

For the purpose of the following result we work with a cofibration $i : A \longrightarrow X$ and we consider the diagram (5.4) below. We also fix a mapping cylinder for i together with a cofibration $h : A \longrightarrow M$, as in Section 4.1. In diagram (5.4) the upper row is the fibration sequence constructed in [5], based on the cofibration $h : A \longrightarrow M$. The lower row of maps is a fibration sequence with p the pinch map. The map $r : F \longrightarrow X \cup CA$ is the inclusion of the homotopy fibre of p. We regard this map r as (having been replaced appropriately by) a fibration. The map γ on the left hand side is the homotopy equivalence constructed in [5]. The map E is to be constructed. Its existence is proved in Theorem 5.2. We note that the solid arrows in Eq. 5.4 constitute a commutative diagram.

$$\begin{array}{ccccccc}
A_\infty & \longrightarrow & (M, A)_\infty & \xrightarrow{\eta} & X \cup CA & & \\
\gamma \downarrow & & \vdots\, \epsilon & & \downarrow = & & \\
\Omega\Sigma A & \longrightarrow & F & \xrightarrow{r} & X \cup CA & \xrightarrow{p} & \Sigma A
\end{array} \qquad (5.4)$$

Theorem 5.2 *Suppose that the model category* **C** *satisfies the necessary conditions from* [5] *to ensure the weak equivalence* $A_\infty \simeq \Omega\Sigma A$.

3. *There exists a map* $E : (M, A)_\infty \rightarrow F$ *making the diagram* (5.4) *commutative.*
4. *If the category* **C** *also satisfies Condition* 3.2, *then* E *is a weak equivalence.*

Proof

(a) Note that $\gamma : A_\infty \rightarrow Q{:}EA$ induces maps $\delta_n : A_n \rightarrow Q{:}EA$. Note also that there is a map $E_1 : M \rightarrow F$ such that $r \circ E_1$ coincides with the restriction $\eta|M$, $\eta|M$ being the composition of the inclusion $M \subset (M, A)_\infty$ followed by η. We observe that $\eta|M = q$, with q as in diagram (4.3).

We note that r is a pull-back of the principle fibration obtained from the map

$$* \longrightarrow \Omega\Sigma A\,,$$

and therefore is itself a principal fibration by Propositions 3.6 and 3.7.

Now given any $n \in \mathbb{N}$, we obtain a map $\xi_{n+1} : M \times A_n \rightarrow F$ as the following composition,

$$M \times A_n \xrightarrow{\epsilon_1 \times \delta_n} F \times \Omega\Sigma A \xrightarrow{\alpha} F\,,$$

where α is the action of the principal fibration, and then $r \circ \zeta_{n+1}$ coincides with the map

$$M \times A_n \xrightarrow{\subset} M \times A_\infty \xrightarrow{\text{proj}} M \xrightarrow{q} X \cup CA\,.$$

$$M \times A_n \xrightarrow{\subset} M \times A_\infty \xrightarrow{\text{proj}} M \xrightarrow{q} X \cup CA\,.$$

$$M \times A_n \xrightarrow{\subset} M \times A_\infty \xrightarrow{\text{proj}} M \xrightarrow{q} X \cup CA\,.$$

In view of the alternative description of objects of the type $(M, A)_{n+1}$ in Proposition 5.1, there exists a map $E_{n+1} : (M, A)_{n+1} \to F$ such that $r \circ E_{n+1}$ coincides with the map induced by η. The sequence of maps (E_n) has a limit, E, which is in fact the map that we want, making the diagram commutative.

(b) Applying Condition 3.2 to diagram (5.4) we conclude that E is a weak equivalence.

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