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Codenseness and Openness with Respect to an Interior Operator

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Abstract
Working in an arbitrary category endowed with a fixed $(\mathcal{E}, \mathcal{M})$-factorization system such that $\mathcal{M}$ is a fixed class of monomorphisms, we first define and study a concept of codense morphisms with respect to a given categorical interior operator i. Some basic properties of these morphisms are discussed. In particular, it is shown that i-codenseness is preserved under both images and dual images under morphisms in $\mathcal{M}$ and $\mathcal{E}$, respectively. We then introduce and investigate a notion of quasi-open morphisms with respect to i. Notably, we obtain a characterization of quasi i-open morphisms in terms of i-codense subobjects. Furthermore, we prove that these morphisms are a generalization of the i-open morphisms that are introduced by Castellini. We show that every morphism which is both i-codense and quasi i-open is actually i-open. Examples in topology and algebra are also provided.

Keywords Interior operator · Codenseness · Openness · Quasi-openness

Mathematics Subject Classification 06A15 · 18A20 · 54B30

1 Introduction

A categorical closure operator on an arbitrary category is a family of functions (on suitably defined subobject lattices) which are expansive, order preserving and compatible with taking images or equivalently, preimages, in the same way as the usual topological closure is compatible with continuous maps. The formal theory of categorical closure operators was introduced by Dikranjan and Giuli [11] and then developed by these authors and Tholen [12]. The theory was largely inspired by Salbany's paper [23], where regular closure operators on the category of topological spaces and continuous maps were introduced. These operators

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have played a vital role in the development of Categorical Topology by introducing topological concepts, such as connectedness, separatedness, compactness, denseness and closedness, in an arbitrary category and they provide a unified approach to various mathematical notions (see [4,13]).

Motivated by the theory of categorical closure operators, the categorical notion of interior operators was introduced by [24]. These operators have received more recent attention and a few papers are published on the subject; see [3,5–9,17,20,22].

In general topology, interior and closure characterize each other via set-theoretic complement. More generally, closure and interior operators characterize each other in a category equipped with a categorical transformation operator (see [24]). As a consequence, most of the theory of interior operators can be derived from that of closure operators and vice versa. Nevertheless, the two operators are not categorically dual to each other on an arbitrary category. It is shown in [24] that the category of groups does not have a categorical transformation, hence the two notions are not necessarily equivalent. Moreover, in any category for which all subobjects are normal, in particular, in all abelian categories (such as the category of modules over a ring, the category of all abelian groups), while there is an abundance of closure operators there is a unique interior operator, which is the discrete one (see [14]).

A crucial property of closure operators is that they are compatible with taking both images and preimages (Captured by the diagonalization lemma in [13]). This plays a significant role in the development of closure operators and enables each closure operator to give rise to an endofunctor of the arrow category $\mathcal{M}$; see [4]. Since categorical interior operators are not compatible with taking images, they cannot be seen as endofunctors on a suitable class $\mathcal{M}$ of embeddings, hence the preservation property of interior operators fails; see [5–7]. Furthermore, the dual closure operator introduced in [14] is a categorical dual to closure operator and does not lead to interior operators.

For the above reasons, the study of a categorical notion of interior operators for its own sake is interesting enough. The aim of this paper is to apply these operators in some areas where they are useful in deriving new results. In particular, we investigate the notions codenseness and quasi-openness with respect to a given interior operator i. Moreover, we provide further characterizations and properties of i-open morphisms.

Most of this paper forms part of the first author's Ph.D. Thesis [2] written under the supervision of the second author.

2 Preliminaries

We use general categorical terminology from [1], while for categorical closure operators we refer to [13] or [4]. Throughout the paper, we consider a finitely complete category $\mathbb{C}$ with a proper $(\mathcal{E}, \mathcal{M})$-factorization system for morphisms (cf. [1]). Consequently, $\mathcal{M}$ is a fixed class of $\mathbb{C}$-monomorphisms and $\mathcal{E}$ is a fixed class of $\mathbb{C}$-epimorphisms, both containing $\mathbb{C}$-isomorphisms, such that each morphism f in $\mathbb{C}$ has an essentially unique factorization given by $f = m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$. Further, $\mathcal{M}$ contains the $\mathbb{C}$-regular monomorphisms and is closed under composition and stable under pullback and is left cancellable.

The category $\mathbb{C}$ is further assumed to be $\mathcal{M}$-complete so that arbitrary $\mathcal{M}$-intersections of $\mathcal{M}$-morphisms exist and belong to $\mathcal{M}$.

For each object $X \in \mathbb{C}$, sub$X$ denotes the class of $\mathcal{M}$-morphisms with codomain X and its objects are known as ($\mathcal{M}$-)subobjects of X. This class is preordered with the relation $r \leq s \Leftrightarrow r = s \circ t$ for some morphism t (which is necessarily unique and in $\mathcal{M}$, and will be denoted by r_s);

$$\begin{array}{ccc} R & \xrightarrow{t=r_s} & S \\ & \searrow^{r} & \downarrow{\scriptstyle s} \\ & & X \end{array}$$

If $r \le s$ and $s \le r$ then r_s is an isomorphism and we write $r \cong s$.

Given any morphism $f: X \to Y$ in $\mathbb{C}$, $m \in \mathrm{sub}X$ and $n \in \mathrm{sub}Y$, the image $f(m)$ of m under f is given by the $\mathcal{M}$-component of the $(\mathcal{E}, \mathcal{M})$-factorization of the composition $f \circ m$. By image of f we mean $f(1_X)$ for $1_X: X \to X$. Preimage $f^{-1}(n)$ of n under f is given by the pullback of n along f.

As a consequence of the above assumptions and terminologies one has:

(a) For each $X \in \mathbb{C}$, $\mathrm{sub}X$ is a complete lattice with $0_X: O_X \to X$ and $1_X: X \to X$ as the least and greatest member of the lattice, respectively. The meet of s and r in $\mathrm{sub}X$ is given by $s \wedge r \cong s \circ s^{-1}(r) \cong r \circ r^{-1}(s)$.

(b) Every morphism $f: X \to Y$ in $\mathbb{C}$ induces an image-preimage adjunction:

$$\mathrm{sub}X \underset{f^{-1}(-)}{\overset{f(-)}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \mathrm{sub}Y$$

that is: $f(m) \le n \Leftrightarrow m \le f^{-1}(n)$ for all $m \in \mathrm{sub}X$ and $n \in \mathrm{sub}Y$.

Furthermore:

Remark 2.1 [10,13] Let $f: X \to Y$ be a morphism in $\mathbb{C}$, $m \in \mathrm{sub}X$ and $n \in \mathrm{sub}Y$. Then:

(a) $f(0_X) \cong 0_Y$, $m \le f^{-1}(f(m))$ and $f(f^{-1}(n)) \le n$.
(b) $f \in \mathcal{E}$ if and only if $f(1_X) \cong 1_Y$.
(c) If $f \in \mathcal{M}$ then $f(m) \cong f \circ m$ and $f^{-1}(f(m)) \cong m$.
(d) Let $\mathcal{E}$ be stable under pullback along $\mathcal{M}$-morphisms. If $f \in \mathcal{E}$ then $f(f^{-1}(n)) \cong n$.

Recall from [17] that a morphism $f: X \to Y$ reflects the least subobject if $f^{-1}(0_Y) \cong 0_X$. Note that each $\mathcal{M}$-subobject reflects the least subobject, since if $f \in \mathcal{M}$ then $f^{-1}(0_Y) \cong f^{-1}(f(0_X)) \cong 0_X$.

We sometimes assume that the preimage $f^{-1}(-): \mathrm{sub}Y \to \mathrm{sub}X$ preserves arbitrary joins for every morphism $f: X \to Y$ in $\mathbb{C}$, as in [2,3,20]. Consequently, $f^{-1}(-)$ has a right adjoint $f_*(-)$, which is given by $f_*(m) = \bigvee\{n \in \mathrm{sub}Y: f^{-1}(n) \le m\}$. Hence, one has $f^{-1}(n) \le m$ if and only if $n \le f_*(m)$ for all $m \in \mathrm{sub}X$ and $n \in \mathrm{sub}Y$. This in turn implies $f^{-1}(0_Y) \cong 0_X$, $f^{-1}(f_*(m)) \le m$ (with "$\cong$" holding true if $f \in \mathcal{M}$) and $n \le f_*(f^{-1}(n))$ (with "$\cong$" holding true if $f \in \mathcal{E}$ with $\mathcal{E}$ is stable under pullback along $\mathcal{M}$-morphisms). We call $f_*(m)$ the dual image of m. In fact, $f_*(m)$ is the greatest subobject $n: N \to Y$ such that the pullback $f^{-1}(n): f^{-1}(N) \to X$ factors through m.

3 Codenseness

We begin with the following definition of an interior operator in an arbitrary category which was introduced by Vorster [24].

Definition 3.1 An interior operator i on $\mathbb{C}$ with respect to $\mathcal{M}$ is a family

$$i = (i_X \colon \mathrm{sub}X \to \mathrm{sub}X)_{X \in \mathbb{C}}$$

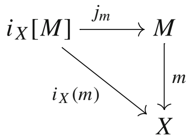
of functions which are

(I_1) contractive: $i_X(m) \leq m$,
(I_2) order preserving: if $k \leq m$ then $i_X(k) \leq i_X(m)$,
(I_3) and which satisfy the continuity condition: $f^{-1}(i_Y(n)) \leq i_X(f^{-1}(n))$,

for all $f \colon X \to Y$ in $\mathbb{C}$ and $k, m \in \mathrm{sub}X$ and $n \in \mathrm{sub}Y$.

In the sequel, unless stated otherwise, we use i to denote an interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$.

Given an $\mathcal{M}$-subobject $m \colon M \to X$, contractivity ensures a unique morphism $j_m \in \mathcal{M}$ such that $i_X(m) = m \circ j_m$.



The following terminologies are from [5,17,24] and will be used whenever necessary.

Definition 3.2 We call

(a) an $\mathcal{M}$-subobject $m \colon M \to X$ i-open (in X) if $i_X(m) \cong m$;
(b) i idempotent if $i_X(i_X(m)) \cong i_X(m)$ for all $m \in \mathrm{sub}X$, $X \in \mathbb{C}$;
(c) i standard if $i_X(1_X) \cong 1_X$ for all $X \in \mathbb{C}$.
(d) i additive if $i_X(m \wedge k) \cong i_X(m) \wedge i_X(k)$ for all $m, k \in \mathrm{sub}X$ and $X \in \mathbb{C}$.

We recall the following definition from [2,3].

Definition 3.3 Assume that for any $\mathbb{C}$-morphism $f : X \to Y$, $f^{-1}(-)$ commutes with the joins of subobjects. Let i be an interior operator on $\mathbb{C}$. Then i is said to be hereditary if for all $r \leq s$ in sub$X$ and $X \in \mathbb{C}$, one has

$$i_S(r_s) \cong s^{-1}(i_X(s_*(r_s))).$$

Now we provide some examples of interior operators from [5,7–9,20,24].

Example 3.4 (a) The discrete interior operator d^{in} is defined by $d^{\mathrm{in}}_X(m) = m$ for all $m \in \mathrm{sub}X$ and $X \in \mathbb{C}$.
(b) Let $\mathbb{C}$ be any category such that all its morphisms reflect the least subobjects (in particular, in categories in which preimages preserve arbitrary joins). Then $t^{\mathrm{in}} = (t^{\mathrm{in}}_X)_{X \in \mathbb{C}}$ with $t^{\mathrm{in}}_X(m) \cong 0_X$ for all $m \in \mathrm{sub}X$ is an interior operator on $\mathbb{C}$. We call t^{in} the trivial interior operator.
(c) Let $\mathbb{C} = \mathbf{Top}$ with (*Surjections*, *Embeddings*)-factorization system and $M \subseteq X \in \mathbf{Top}$. The assignments

(i) $k^{\mathrm{in}}_X(M) = \bigcup\{O \text{ open in } X \colon O \subseteq M\}$,
(ii) $k^{*\mathrm{in}}_X(M) = \bigcup\{C \text{ closed in } X \colon C \subseteq M\}$, and
(iii) $q^{\mathrm{in}}_X(M) = \bigcup\{O \text{ clopen in } X \colon O \subseteq M\}$

define standard, additive and idempotent interior operators on **Top** with respect to embeddings. The operators k^{in}, $k^{*\text{in}}$, and q^{in} are called the Kuratowski, inverse Kuratowski, and quasicomponent interior operators, respectively. The k^{in} ($k^{*\text{in}}$, q^{in}, resp.)-open subspaces of a topological space X are exactly the open (closed, clopen, resp.) subsets of X.

(d) Let $\mathbb{C} = \textbf{Grp}$ with the *(Surjective homomorphisms, Injective homomorphisms)*-factorization system and $H \leq G \in \textbf{Grp}$. $n_G(H) = \bigvee\{N \trianglelefteq G : N \leq H\}$ defines a standard, additive and idempotent interior operator of **Grp** called the normal interior operator. The n-open subgroups of a group G are precisely the normal subgroups of G. We remark that G is a Dedekind group if and only if the normal and discrete interior operators coincide.

(e) Consider $\mathbb{C} = \textbf{Rng}$ with the (*Surjective homomorphisms, Injective homomorphisms*)-factorization system and $S \leq R \in \textbf{Rng}$. Then the assignment $j_R(S) = \bigvee\{I \trianglelefteq R : I \leq S\}$ is a standard, additive and idempotent interior operator of **Rng**. $J = (j_R)_{R \in \mathbb{C}}$ is called the ideal interior operator. The J-open subrings of a ring R are precisely the ideals of R. We note that even if R has a unity, its subrings need not contain this. Let us also note that if R is a cyclic ring (a ring in which its additive group is cyclic, hence each of its subrings is an ideal), then the ideal and discrete interior operators coincide.

Definitions 3.1 and 3.2(a) allow us to obtain:

Remark 3.5 The preimage of an i-open subobject is an i-open subobject; see [5–7].

Next we introduce some basic properties of the notion of codenseness with respect to an interior operator i on $\mathbb{C}$ with respect to $\mathcal{M}$. To this end, we first observe the following:

Remark 3.6 Recall from [15] that a subset M of a topological space X is called codense in X if the complement $X \backslash M$ of M in X is dense in X, that is: if $k_X(X \backslash M) = X$, which is equivalent to $k_X^{\text{in}}(M) = \emptyset$, where k and k^{in} are the Kuratowski closure and interior operators, respectively.

The above description of codenseness in terms of the Kuratowski interior operator motivates the following:

Definition 3.7 Given an interior operator i, we say that an $\mathcal{M}$-subobject $m : M \to X$ is i-codense (also called i-isolated in [5,9]) in X if $i_X(m) \cong 0_X$.

A notion of i-codense subobjects was used in [9] to define indiscrete objects with respect to an interior operator i in order to introduce the notion of disconnectedness in **Top**.

Remark 3.8 (a) A subobject of an object $X \in \mathbb{C}$ is both i-codense and i-open if and only if it is the least subobject of X.

(b) For a standard interior operator i, the identity morphism $1_X : X \to X$ on a non-trivial object $X \in \mathbb{C}$ can not be i-codense. Note that an object X of $\mathbb{C}$ is trivial if $0_X \cong 1_X$.

(c) Let $m \leq k$ be subobjects of X. If k is i-codense in X, so is m.

Example 3.9 (a) Consider the normal interior operator n on **Grp**. Then for any group G, the n-codense subgroups are precisely the subgroups $H \neq G$ which do not contain proper normal subgroups of G.

(b) Consider the ideal interior operator J on **Rng**. Then every subring $S \neq R$ of a simple ring (a non-zero ring that has no proper ideal) R is J-codense in R while (0_R) is the only J-codense subring of a cyclic ring R.

The following property of $\mathcal{M}$-morphisms will be useful.

Lemma 3.10 *Let i be an interior operator on $\mathbb{C}$ with respect to $\mathcal{M}$ and let $f\colon X \to Y$ be a morphism in $\mathcal{M}$. Then $i_Y(f(m)) \leq f(i_X(m))$ for all $m \in subX$.*

Proof Since $f(m) \cong f \circ m \leq f$, one has $i_Y(f(m)) \leq i_Y(f) \leq f$. Consequently, $i_Y(f(m)) \cong f \wedge i_Y(f(m)) \cong f \circ f^{-1}(i_Y(f(m))) \leq f \circ i_X(f^{-1}(f(m))) \cong f \circ i_X(m) \cong f(i_X(m))$. □

Consequently, i-codenseness is preserved by images under $\mathcal{M}$-morphisms:

Proposition 3.11 *Let $f\colon X \to Y$ be a morphism in $\mathbb{C}$, $m \in subX$ and $n \in subY$.*

(a) *If $f \in \mathcal{M}$ and m is i-codense in X, then $f(m)$ is i-codense in Y.*
(b) *If $f \in \mathcal{E}$ with $\mathcal{E}$ stable under pullback along $\mathcal{M}$-morphisms and $f^{-1}(n)$ is i-codense in X, then n is i-codense in Y.*

Proof (a) Indeed, since $f \in \mathcal{M}$, one has $i_Y(f(m)) \leq f(i_X(m))$ by Lemma 3.10, hence $i_Y(f(m)) \leq f(i_X(m)) \cong f(0_X) \cong 0_Y$.
(b) If $i_X(f^{-1}(n)) \cong 0_X$ and $f \in \mathcal{E}$, then $i_Y(n) \cong f(f^{-1}(i_Y(n))) \leq f(i_X(f^{-1}(n))) \cong f(0_X) \cong 0_Y$. □

Remark 3.12 Let preimages commute with arbitrary joins in the category $\mathbb{C}$, $f\colon X \to Y$ a morphism in $\mathbb{C}$, $m \in \mathrm{sub}X$ and $n \in \mathrm{sub}Y$. Since $f^{-1}(f_*(m)) \leq m$ and $n \leq f_*(f^{-1}(n))$, it follows from Remark 3.8(c) that the statements

(i) if $f^{-1}(n)$ is i-codense in X, then n is i-codense in Y,
(ii) if m is i-codense in X, then $f_*(m)$ is i-codense in Y

are equivalent. Consequently, Proposition 3.11(b) yields that i-codenseness is preserved by dual images under $\mathcal{E}$-morphisms whenever $\mathcal{E}$ is stable under pullback along $\mathcal{M}$-morphisms, that is: if $f \in \mathcal{E}$ then (ii) holds.

Codenseness is naturally extended from subobjects to i-codense morphisms.

Definition 3.13 We say that a $\mathbb{C}$-morphism $f\colon X \to Y$ is i-codense whenever its image $f(1_X)$ is an i-codense subobject of Y, that is: $i_Y(f(1_X)) \cong 0_Y$. We denote by $\mathcal{CD}(i)$ the class of i-codense morphisms in $\mathbb{C}$.

The following characterization is then an immediate consequence of Remark 3.8 and Definition 3.13.

Proposition 3.14 *The following statements are equivalent for a $\mathbb{C}$-morphism $f\colon X \to Y$:*

(a) *$f \in \mathcal{CD}(i)$;*
(b) *For all $m \in subX$, $f(m)$ is an i-codense subobject of Y.*

We can now provide some of the stability properties of $\mathcal{CD}(i)$.

Proposition 3.15 *The class $\mathcal{CD}(i)$*

(a) *is stable under composition with $\mathbb{C}$-morphisms from the right, that is: if $g \in \mathcal{CD}(i)$ and f in $\mathbb{C}$ then $g \circ f \in \mathcal{CD}(i)$,*
(b) *is stable under composition with $\mathcal{M}$-morphisms from the left, that is: if $g \in \mathcal{M}$ and $f \in \mathcal{CD}(i)$ then $g \circ f \in \mathcal{CD}(i)$, and*
(c) *is right-cancellable with respect to $\mathcal{E}$, that is: if $g \circ f \in \mathcal{CD}(i)$ and $f \in \mathcal{E}$ then $g \in \mathcal{CD}(i)$.*

4 Openness

The notion of an open morphism with respect to an interior operator was introduced in [6]. We investigate this further, giving a number of new characterizations and some properties of i-open morphisms.

Definition 4.1 [6] A morphism $f: X \to Y$ is called i-open if $f(i_X(m)) \le i_Y(f(m))$ for all $m \in \text{sub}X$. We denote by $\mathcal{O}(i)$ the class of i-open morphisms in $\mathbb{C}$.

Since, by adjointness, $f(i_X(m)) \le i_Y(f(m)) \Leftrightarrow i_X(m) \le f^{-1}(i_Y(f(m)))$ for all $m \in \text{sub}X$, one has $f: X \to Y \in \mathcal{O}(i) \Leftrightarrow i_X(m) \cong i_X(m) \wedge f^{-1}(i_Y(f(m)))$ for all $m \in \text{sub}X$. Moreover, from the order preservation and continuity condition of i and from the adjointness property one obtains:

Proposition 4.2 *For a morphism $f: X \to Y$ in $\mathbb{C}$, the following statements are equivalent:*

(a) $f \in \mathcal{O}(i)$;
(b) $m \le i_X(f^{-1}(n)) \Leftrightarrow f(m) \le i_Y(n)$ *for all* $m \in \text{sub}X$ *and* $n \in \text{sub}Y$;
(c) $i_X(f^{-1}(n)) \cong f^{-1}(i_Y(n))$ *for all* $n \in \text{sub}Y$ (see [6]).

Proposition 4.3 [6] *Let O^i_X denote the class of i-open subobjects of X. If $f: X \to Y \in \mathcal{O}(i)$ then $\left(m \in O^i_X \Rightarrow f(m) \in O^i_Y\right)$ for all $m \in \text{sub}X$. Moreover, if i is idempotent then the converse is true.*

Remark 4.4 (a) Let $f: X \to Y \in \mathcal{O}(i)$ and let $m \in O^i_X$. Then the above proposition implies that $f(m) \in O^i_Y$. Consequently, by Remark 3.5, $f^{-1}(f(m)) \in O^i_X$.
(b) For a standard interior operator i, since 1_X is an i-open subobject, the image of an i-open morphism is i-open by Proposition 4.3.

Proposition 4.3 states that for an idempotent interior operator i, i-open morphisms are characterized by preservation of i-open subobjects. Consequently,

Example 4.5 (a) In **Top**, the open morphisms with respect to the inverse Kuratowski (quasicomponent, resp.) interior operator are precisely the continuous maps which preserve closed (clopen, resp.) subspaces since both operators are idempotent.
(b) In **Grp**, since the normal interior operator n is idempotent, the open morphisms with respect to n are precisely the group homomorphims which preserve normal subroups. Consequently, any surjective group homomorphism is an n-open morphism.
(c) In **Rng**, since the ideal interior operator J is idempotent, the open morphisms with respect to J are exactly the ring homomorphims which preserve ideals. Consequently, any surjective ring homomorphism is a J-open morphism.

Corollary 4.6 *Let $r_s: R \to S$ be an i-open $\mathcal{M}$-subobject, and let $s: S \to X$ be an i-open $\mathcal{M}$-morphism. Then $r = s \circ r_s$ is i-open $\mathcal{M}$-subobject of X.*

Proof This follows immediately from Remark 2.1(c) and Proposition 4.3. □

As a consequence of Lemma 3.10, Definition 4.1 and Proposition 4.3, one has:

Proposition 4.7 *Let $f: X \to Y$ be an $\mathcal{M}$-morphism. Then the statements (a), (b), (c) below are equivalent and imply (d):*

(a) $f \in \mathcal{O}(i)$,

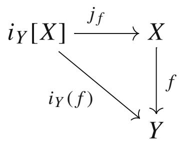
(b) $f(i_X(m)) \cong i_Y(f(m))$ *for all* $m \in subX$,
(c) $i_X(m) \cong f^{-1}(i_Y(f(m)))$ *for all* $m \in subX$,
(d) $(\forall m \in subX)\ \big(m \in O^i_X \Leftrightarrow f(m) \in O^i_Y\big)$.

All four statements are equivalent if i *is idempotent.*

From the above proposition we conclude that i-open $\mathcal{M}$-morphisms are precisely the morphisms whose images commute with the interior i. Moreover,

Proposition 4.8 *The following statements hold for an* i*-open* $\mathcal{M}$*-morphism* $f: X \to Y$*:*

(a) *Every* i*-open subobject* m *of* X *is of the form* $f^{-1}(n)$ *for some* i*-open subobject* n *of* Y*;*
(b) *If* i *is idempotent then* $j_f: i_Y[X] \to X$ *is an* i*-open subobject* (*see* [6]).



The next proposition characterizes i-openness for $\mathcal{E}$-morphisms.

Proposition 4.9 *Let* $f: X \to Y$ *be a morphism in* $\mathcal{E}$ *with* $\mathcal{E}$ *stable under pullback along* $\mathcal{M}$*-morphisms. Then the conditions* (a), (b) *below are equivalent and imply* (c)*:*

(a) $f \in \mathcal{O}(i)$,
(b) $i_Y(n) \cong f(i_X(f^{-1}(n)))$ *for all* $n \in subY$,
(c) $(\forall n \in subY)\ \big(n \in O^i_Y \Leftrightarrow f^{-1}(n) \in O^i_X\big)$.

Proof $(a) \Leftrightarrow (b)$ follows from Proposition 4.2 and the adjointness property.
$(a) \Rightarrow (c)$ Let $n \in O^i_Y$. Then Remark 3.5 implies $f^{-1}(n) \in O^i_X$. On the other hand, suppose $f^{-1}(n) \in O^i_X$. Then Proposition 4.3 and Remark 2.1(d) imply $n \cong f(f^{-1}(n)) \in O^i_Y$ since $f \in \mathcal{E}$.

□

The following is a characterization of open morphisms with respect to any idempotent interior operator.

Proposition 4.10 *Let* i *be idempotent. A morphism* $f: X \to Y \in \mathcal{O}(i)$ *if and only if for every* i*-open subobject* m *of* X *and for every subobject* n *of* Y *such that* $m \leq f^{-1}(n)$*, there exists an* i*-open subobject* k *of* Y *such that* $k \leq n$ *and* $m \leq f^{-1}(k)$.

Proof

($\Rightarrow$) Suppose $f : X \to Y \in \mathcal{O}(i)$, $n \in \text{sub}Y$ and m is i-open subobject of X such that $m \leq f^{-1}(n)$. Then, there exists $k = f(m)$ such that $k = f(m) \leq n$ and $k = f(m)$ is i-open subobject of Y by Proposition 4.3(b). Moreover, $m \leq f^{-1}(f(m)) = f^{-1}(k)$.
($\Leftarrow$) Suppose $f: X \to Y$ satisfies the condition in the proposition and let m be an i-open subobject of X. Then, for $n = f(m)$, one has $m \leq f^{-1}(f(m)) = f^{-1}(n)$. Consequently, there exists an i-open subobject k of Y such that $k \leq f(m) = n$ and $m \leq f^{-1}(k)] \Leftrightarrow [k \leq f(m)$ and $n = f(m) \leq k]$. As a result, $f(m) \cong k$ and hence $f(m)$ is an i-open subobject of Y. Therefore, by Proposition 4.3(b), f is i-open.

□

Remark 4.11 Let $m, n \in \mathcal{M}$, $f \in \mathcal{O}(i)$ and the following commute.

$$\begin{array}{ccc} M & \xrightarrow{u} & N \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ X & \xrightarrow[f]{} & Y \end{array}$$

(a) [6,7]There is a uniquely determined morphism $w: i_X[M] \to i_X[N]$ making the diagram

$$\begin{array}{ccc} i_X[M] & \xrightarrow{w} & i_X[N] \\ {\scriptstyle j_m}\downarrow & & \downarrow{\scriptstyle j_n} \\ M & \xrightarrow{u} & N \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutative, that is: the preservation property of i holds if and only if $f \in \mathcal{O}(i)$.

(b) If n is i-open subobject, then there is a uniquely determined morphism $s: i_X[M] \to N$ with $s = 1_N \circ s = u \circ j_m$ and $n \circ s = f \circ i_X(m)$.

(c) If m is i-codense subobject, then there is a uniquely determined morphism $t : O_M \to i_X[N]$ with $u \circ 0_M = j_m \circ t$ and $f \circ 0_X = i_X(m)$.

(d) If m is i-codense subobject and n is i-open subobject, then there is a uniquely determined morphism $d : O_M \to N$ with $u \circ 0_M = d$ and $n \circ d = f \circ 0_X$.

Indeed, (b), (c) and (d) are direct consequences of (a).

For the remainder of this section we assume that the class $\mathcal{E}$ is stable under pullback along $\mathcal{M}$-morphisms and for any $\mathbb{C}$-morphism $f: X \to Y$, $f^{-1}(-)$ commutes with the joins of subobjects.

Definition 4.12 [2,3,20] Let i be an interior operator on $\mathbb{C}$. A $\mathbb{C}$-morphism $f: X \to Y$ is called

(a) i-initial if $i_X(m) \cong f^{-1}(i_Y(f_*(m)))$ for all $m \in \text{sub}X$;

(b) i-final if $f_*(i_X(f^{-1}(n))) \cong i_Y(n)$ for all $n \in \text{sub}Y$.

Now one can easily prove the following connection of i-open morphisms with i-initial and i-final morphisms.

Proposition 4.13 *Let i be an interior operator on $\mathbb{C}$. Then:*

(a) *An i-open $\mathcal{M}$-morphism is i-initial;*

(b) *An i-open $\mathcal{E}$-morphism is i-final;*

(c) *An i-initial $\mathcal{E}$-morphism is i-open;*

(d) *An i-final $\mathcal{M}$-morphism is i-open.*

5 Quasi-openness

We now want to introduce a notion of quasi-open morphisms with respect to an interior operator i on an arbitrary category $\mathbb{C}$ and discuss some of their properties. In particular, it is shown that the quasi i-open morphisms of $\mathbb{C}$ are characterized as the morphisms which reflect i-codensity. To this end, we begin with the following definition and proposition which motivate the notion of quasi i-open morphisms.

Definition 5.1 A morphism $f: X \to Y$ is said to reflect i-codensity if $f^{-1}(-)$ maps i-codense subobjects of Y to i-codense subobjects of X.

Consequently,

Proposition 5.2 *Suppose $f: X \to Y \in \mathcal{O}(i)$ reflects the least subobject. Then f reflects i-codensity.*

Proof Let n be an i-codense subobject of Y. Then, $i_Y(n) \cong 0_Y$. Consequently, $i_X(f^{-1}(n)) \cong f^{-1}(i_Y(n)) \cong f^{-1}(0_Y) \cong 0_X$ since f is an i-open morphism and reflects 0_Y. □

Note that in any category in which the preimage functor for any given morphism preserves arbitrary joins (in particular, in topological categories $\mathbb{C}$ over **Set**), each morphism reflects the least subobject. Consequently, with the above proposition one has:

Proposition 5.3 *Assume that the class $\mathcal{E}$ is stable under pullback along $\mathcal{M}$-morphisms and for any $\mathbb{C}$-morphism $f : X \to Y$, $f^{-1}(-)$ commutes with the joins of subobjects. Let i be a hereditary interior operator on $\mathbb{C}$. Then i-open morphisms are stable under pullback along $\mathcal{M}$-morphisms (see [2,3]) and reflect i-codensity.*

Remark 5.4 (a) Let sub$X$ be a Boolean algebra for every $\mathbb{C}$-object X and suppose complements are preserved by preimages. Let c be a closure operator and i^c be the induced interior operator from c given by $i^c_X(m) = \overline{c_X(\overline{m})}$ for all $m \in$ sub$X$, where $\overline{m}$ denotes the complement of m. Then an $\mathcal{M}$-subobject $r : R \to X$ is i^c-codense in X if and only if $\overline{r}$ is c-dense in X. Consequently, a $\mathbb{C}$-morphism f reflects i^c-codensity if and only if it reflects c-density.

(b) A morphism which reflects i-codensity need not be i-open. Indeed, in **Top** the embedding $r: [0, 1] \hookrightarrow \Re$ reflects codensity with respect to the Kuratowski interior operator k^{in} induced by the Euclidean topology but r is not k^{in}-open map.

While any i-open morphism which reflects the least subobject reflects i-codensity, a morphism which reflects i-codensity may not be i-open by Remark 5.4(b). This motivates the following:

Definition 5.5 A morphism $f: X \to Y$ is said to be quasi i-open if the interior of each subobject of X is the least subobject of X whenever the interior of its image under f is the least subobject of Y, that is: $(\forall m \in \text{sub}X)\ (i_Y(f(m)) \cong 0_Y \Rightarrow i_X(m) \cong 0_X)$.

In **Top**, the quasi open morphisms with respect to the Kuratowski interior operator k^{in} are precisely the quasi open maps studied in [18,19,21]. Such maps are also called semi-open in [16].

The following is a handy characterization of quasi i-open morphisms in terms of i-codensity.

Proposition 5.6 *For a morphism $f: X \to Y$ in $\mathbb{C}$, the following are equivalent:*

(a) *f is quasi i-open;*

(b) *each subobject of X is i-codense in X whenever its image under f is i-codense in Y, that is: $(\forall m \in subX)(f(m)$ is i-codense in $Y \Rightarrow m$ is i-codense in $X)$;*

(*c*) *f reflects i-codensity, that is: if n is i-codense in Y then* $f^{-1}(n)$ *is i-codense in X.*

Proof $(a) \Rightarrow (b)$ follows immediately from the definitions.

$(b) \Rightarrow (c)$ Let n be a i-codense in Y. Since $f(f^{-1}(n)) \leq n$, one has $f(f^{-1}(n))$ is i-codense in Y by Remark 3.8(c). Consequently, $f^{-1}(n)$ is i-codense in X.

$(c) \Rightarrow (a)$ Let $m \in \text{sub}X$ such that $i_Y(f(m)) \cong 0_Y$. Then $f(m)$ is i-codense in Y, hence $f^{-1}(f(m))$ is i-codense in X by hypothesis. Consequently, by Remark 3.8(c), m is i-codense in X since $m \leq f^{-1}(f(m))$. Therefore, $i_X(m) \cong 0_X$. □

The above proposition states that the quasi i-open morphisms of $\mathbb{C}$ are precisely the morphisms which reflect i-codensity. Consequently, with Proposition 5.2, quasi i-open morphisms are a generalization of i-open morphisms.

As a consequence of Propositions 3.11 and 5.6 the following corollaries are now evident.

Corollary 5.7 *The following statements are equivalent for an* $\mathcal{M}$*-morphism* $f: X \to Y$*:*

(a) *f is quasi i-open;*
(b) $(\forall m \in subX)$ $(f(m)$ *is i-codense in Y* $\Leftrightarrow$ *m is i-codense in X*$)$

Corollary 5.8 *Let* $\mathcal{E}$ *be stable under pullback along* $\mathcal{M}$*-morphisms. Then the following statements are equivalent for an* $\mathcal{E}$*-morphism* $f: X \to Y$*:*

(*a*) *f is quasi i-open;*
(*b*) $(\forall n \in subY)$ $\big(f^{-1}(n)$ *is i-codense in X* $\Leftrightarrow$ *n is i-codense in Y*$\big)$

Proposition 5.9 *Every i-open morphism in the class* $\mathcal{M}$ *is quasi i-open.*

Proof This follows from Propositions 5.2 and 5.6 since each subobject morphism reflects the least subobject. □

We use $\mathcal{QO}(i)$ to denote the class of quasi i-open morphisms, and C^i the i-codense $\mathcal{M}$-subobjects.

Proposition 5.10 (a) C^i *is stable under pullback along* $\mathcal{QO}(i)$*-morphisms.*
(b) C^i *is left-cancellable with respect to the class of i-open morphisms in* $\mathcal{M}$*.*
(c) $\mathcal{CD}(i)$ *is left-cancellable with respect to* $\mathcal{QO}(i)$*, that is: if* $g \circ f \in \mathcal{CD}(i)$ *and* $g \in \mathcal{QO}(i)$ *then* $f \in \mathcal{CD}(i)$*.*

Proof (b) Indeed, for $s, t \in \mathcal{M}$ such that $s \circ t \in C^i$ and $s \in \mathcal{O}(i)$, one has the pullback diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{1} & \cdot \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s \circ t} \\ \cdot & \xrightarrow[s]{} & \cdot \end{array}$$

with $s \in \mathcal{QO}(i)$ by Remark 5.9. Consequently, by (a), $t \in C^i$ since t is a pullback of $s \circ t \in C^i$ along $s \in \mathcal{QO}(i)$.

(c) Let $f: X \to Y$ be any morphism in $\mathbb{C}$ and $g: Y \to Z \in \mathcal{QO}(i)$ such that $g \circ f \in \mathcal{CD}(i)$. Then $g(f(1_X)) \cong (g \circ f)(1_X)$ is an i-codense subobject of Z. Hence, by Proposition 5.6, $g^{-1}(g(f(1_X)))$ is an i-codense subobject of Y. Consequently, by Remark 3.8(c), $f(1_X)$ is an i-codense subobject of Y since $f(1_X) \leq g^{-1}(g(f(1_X)))$ (see Remark 2.1(a)). Therefore, $f \in \mathcal{CD}(i)$. □

The following property of an i-open subobject will be employed sometimes.

Proposition 5.11 [2,3] *Assume that for any $\mathbb{C}$-morphism $f: X \to Y$, $f^{-1}(-)$ commutes with the joins of subobjects. Let i be an additive and hereditary interior operator on $\mathbb{C}$. If $s: S \to X$ is an i-open subobject of $X \in \mathbb{C}$ then it is an i-open morphism.*

Now we are ready to conclude:

Corollary 5.12 *Assume that for any $\mathbb{C}$-morphism $f : X \to Y$, $f^{-1}(-)$ commutes with the joins of subobjects. Let i be an additive and hereditary interior operator $\mathbb{C}$. Then the class of i-codense $\mathcal{M}$-subobjects is left-cancellable with respect to the class of i-open subobjects.*

Proposition 5.13 *Every quasi i-open morphism reflects the least subobject.*

Proof Let $f: X \to Y \in \mathcal{QO}(i)$. Since 0_Y is both i-open and i-codense, one has $f^{-1}(0_Y)$ is both i-open in X (see Remark 3.5) and i-codense (see Proposition 5.6). Hence, by Remark 3.8(a), $f^{-1}(0_Y)$ is the least subobject of X, that is $f^{-1}(0_Y) \cong 0_X$. Therefore, f reflects the least subobject. □

The above proposition, in turn, implies that the kernel of a quasi i-open morphism in the category **Grp** (**Rng**, resp.) is the trivial subgroup (subring, resp.). Consequently, a quasi i-open morphism in **Grp** (**Rng**, resp.) must be an injective group (ring, resp.) homomorphism.

A quasi i-open morphism generally fails to be i-open (see Proposition 5.6, Remark 5.4(b)). This motivates the following result.

Proposition 5.14 *A morphism which is both i-codense and quasi i-open is i-open.*

Proof Let $f : X \to Y \in \mathcal{CD}(i) \cap \mathcal{QO}(i)$. Since f is an i-codense morphism, then by Proposition 3.14, $f(m)$ is i-codense in Y, that is $i_Y(f(m)) \cong 0_Y$ for all $m \in \text{sub}X$. Consequently, since f is a quasi i-open morphism, m is i-codense in X, that is $i_X(m) \cong 0_X$ for all $m \in \text{sub}X$ (see Proposition 5.6). Hence, $f(i_X(m)) \cong f(0_X) \cong 0_Y \cong i_Y(f(m))$ for all $m \in \text{sub}X$. Therefore, f is an i-open morphism by Definition 4.1. □

The class $\mathcal{QO}(i)$ of quasi i-open morphisms satisfies properties similar to the class $\mathcal{O}(i)$ of i-open morphims given in [6]:

Proposition 5.15 *The class $\mathcal{QO}(i)$*

(a) *contains all the isomorphisms and is closed under composition,*
(b) *is left-cancellable with respect to $\mathcal{M}$ and*
(c) *is right-cancellable with respect to $\mathcal{E}$ if $\mathcal{E}$ is stable under pullback along $\mathcal{M}$-morphisms.*

Proof Consider the morphisms $f: X \to Y$ and $g: Y \to Z$ in $\mathbb{C}$.

(a) Suppose $f, g \in \mathcal{QO}(i)$. Let $m \in \text{sub}X$ such that $(g \circ f)(m)$ is i-codense in Z. Then $g(f(m))$ is i-codense in Z since $(g \circ f)(m) \cong g(f(m))$. Consequently, by Proposition 5.6, $f(m)$ is i-codense in Y. This in turn implies m is i-codense in X by Proposition 5.6. Therefore, $g \circ f \in \mathcal{QO}(i)$.
(b) Suppose $g \circ f \in \mathcal{QO}(i)$ and $g \in \mathcal{M}$. Let $m \in \text{sub}X$ such that $f(m)$ is i-codense in Y. Then by Proposition 3.11(a) $g(f(m))$ is i-codense in Z. Hence, $(g \circ f)(m)$ is i-codense in Z since $(g \circ f)(m) \cong g(f(m))$. Thus, by Proposition 5.6, m is i-codense in X. Therefore, $f \in \mathcal{QO}(i)$.

(c) Suppose $g \circ f \in \mathcal{QO}(i)$ and $f \in \mathcal{E}$. Let $n \in \text{sub}Y$ such that $g(n)$ is i-codense in z. Since $f \in \mathcal{E}$, one has $(g \circ f)(f^{-1}(n)) \cong g(f(f^{-1}(n))) \cong g(n)$ is i-codense in z. Consequently, by Proposition 5.6, $f^{-1}(n)$ is i-codense in X. Hence, by Proposition 3.11(b), n is i-codense in$Y$. Therefore, $g \in \mathcal{QO}(i)$.

□

As an immediate consequence of the above proposition, one has:

Corollary 5.16 *Let $f = m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$ and $\mathcal{E}$ stable under pullback along $\mathcal{M}$-morphisms. $f \in \mathcal{QO}(i)$ if and only if $m, e \in \mathcal{QO}(i)$.*

Corollary 5.17 *Assume that for any $\mathbb{C}$-morphism $f: X \to Y$, $f^{-1}(-)$ commutes with the joins of subobjects. Let i be an additive and hereditary interior operator on $\mathbb{C}$ and let $f: X \to Y \in \mathcal{QO}(i)$. If $m: M \to X$ is an i-open subobject of X then $f \circ m \in \mathcal{QO}(i)$.*

One can drop the additivity and heredity conditions in the above proposition if i is standard and m is an i-open morphism (see [6]).

From Proposition 5.15 and properties of pullbacks, the class $\mathcal{QO}(i)^*$ of stably quasi i-open morphisms satisfies the following fundamental stability properties. Note that by a stably quasi i-open morphism we mean that every pullback of the morphism is quasi i-open.

Proposition 5.18 *The class $\mathcal{QO}(i)^*$*

(a) *contains all the isomorphisms and is closed under composition,*
(b) *is left-cancellable with respect to $\mathcal{M}$, that is: if $g \circ f \in \mathcal{QO}(i)$ and $g \in \mathcal{M}$ then $f \in \mathcal{QO}(i)$ and*
(c) *is right-cancellable with respect to $\mathcal{E}$ if $\mathcal{E}$ is stable under pullback, that is: if $g \circ f \in \mathcal{QO}(i)$ and $f \in \mathcal{E}$ then $g \in \mathcal{QO}(i)$.*

As an application of Propositions 5.2, 5.6, Remark 5.9 and Proposition 5.13, we obtain:

Example 5.19 (a) In **Top**:

(i) Every map which preserves open (closed, clopen, resp.) subspaces is quasi k^{in} ($k^{*\text{in}}$, q^{in})-open;
(ii) Every proper (= stably-closed) map is quasi $k^{*\text{in}}$-open;
(iii) $[0, 1] \hookrightarrow \mathbb{R}$ is quasi k^{in}-open but $\mathbb{Q} \hookrightarrow \mathbb{R}$ is not quasi k^{in}-open.

(b) Each i-open morphism in any category in which preimages preserve arbitrary joins (in particular, in topological categories $\mathbb{C}$ over **Set**) is quasi i-open.
(c) Every i-open $\mathcal{M}$-morphism in an arbitrary category $\mathbb{C}$ is quasi i-open. In particular:

(i) Any injective group homomorphism that preserves normal subgroups is quasi-open with respect to the normal interior operator on **Grp**.
(ii) Any injective ring homomorphism that preserves ideals is quasi-open with respect to the ideal interior operator on **Rng**.

(d) Any injective group homomorphism $f: S \to G$, where S is a simple group and G is a group with $f(S)$ containing a proper normal subgroup of G is quasi-open with respect to the normal interior operator n on **Grp**.

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