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Relative Homotopy in Relational Structures

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Abstract. The homotopy groups of a finite partially ordered set (poset) can be described entirely in the context of posets, as shown in a paper by B. Larose and C. Tardif. In this paper we describe the relative version of such a homotopy theory, for pairs (X, A) where X is a poset and A is a subposet of X. We also prove some theorems on the relevant version of the notion of weak homotopy equivalences for maps of pairs of such objects. We work in the category of reflexive binary relational structures which contains the posets as in the work of Larose and Tardif.

1 Introduction

It is well known that a *partial order* (a reflexive, transitive and anti-symmetric relation) on a set X determines a T_0 topology on X which has as a basis the collection $\{U_x : x \in X\}$ where, for each $x \in X$, $U_x = \{y \in X : y \leq x\}$. This assignment is a functor from the category $\mathcal{P}$ of *posets* (partially ordered sets) and order-preserving functions to the category of T_0 topological spaces and continuous functions. In fact, this functor establishes an isomorphism between the subcategories of finite posets and finite T_0 spaces.

1.1 Notation

For a poset X, let $\mathcal{U}(X) = \{U_x : x \in X\}$ where, for each $x \in X$,

$$U_x = \{y \in X : y \leq x\}.$$

On the other hand, there is also a link between finite posets and finite polyhedra via a functor $\mathcal{K}$ defined by Alexandroff [2]. For a finite poset X, $\mathcal{K}X$ is the simplicial complex whose vertices are the points of X and whose simplexes are the totally ordered subsets of X. McCord [10] showed that the associated T_0-space X had exactly the same homology and homotopy groups as the underlying polyhedron $|\mathcal{K}X|$. Specifically, he defined a natural transformation $qX\colon |\mathcal{K}X| \to X$ which he proved to be a weak homotopy equivalence if X is an A-space (a topological space in which the intersection of every collection of open sets is open) and in particular if X is *locally finite* (a space in which every point has a finite closure and a finite neighbourhood).

What this means is that the homomorphisms of homotopy groups (for $n \geq 1$) and the function (for $n = 0$) $(qX)_*\colon \pi_n|\mathcal{K}X| \to \pi_n(X)$, induced by qX are isomorphisms for $n \in \mathbb{N}$ and a bijection for $n = 0$. Following this connection, examples of

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finite space models of certain spaces and maps between them have been described, for instance in [6, 7].

Very recently, Larose and Tardif [8] have constructed an *internal* version of the homotopy groups of a poset X. They define analogues of homotopy groups as certain equivalence classes of maps between posets. Actually, they work in a category which is bigger than $\mathcal{P}$.

1.2 Relational Structure

A *binary reflexive relational structure* $\mathbf{X} = (X, \theta)$ is a set X together with a reflexive binary relation $\theta \subseteq X \times X$. The category $\mathcal{R}$ is the category of which the objects are the binary reflexive relational structures and the morphisms are the functions $f\colon (X, \theta_X) \to (Y, \theta_Y)$ satisfying the condition $(x_1, x_2) \in \theta_X \Rightarrow (fx_1, fx_2) \in \theta_Y$.

The reflexivity condition, *i.e.,* that $(x, x) \in \theta_X$ for all $x \in X$, implies in particular that every *constant* function between relational structures is an $\mathcal{R}$-morphism. The category $\mathcal{R}$ is a *construct*, see [1, 5.1, p. 53; 3.3, p. 14], *i.e.,* its objects are structured sets and the morphisms are structure preserving functions. For an object (X, θ_X) of $\mathcal{R}$, by a *regular subobject* of (X, θ_X) we mean an object (Y, θ_Y) of $\mathcal{R}$ such that $Y \subseteq X$ and $\theta_Y = \{(x, y) \in \theta_X : x, y \in Y\}$. Henceforth we shall simply say *subobject*.

In the sequel the term binary reflexive relational structure will be shortened to *relational structure*. Furthermore we shall refer to a relational structure (X, θ) as X, without explicitly specifying the relation θ when there is no ambiguity.

For decades, posets have been utilized in the study of cohomology of groups. Finite models of sphere multiplications and Hopf constructions appear in [6, 7]. Relational structures find application in theoretical computer science. These applications include complexity problems such as in [4, 8], for instance.

The definition in [8] is reminiscent of the classical definition of the homotopy groups $\pi_n(Y, *)$ of the space Y. The group $\pi_n(Y, *)$ consists of homotopy classes of maps from the n-fold unit cube I^n into Y which send the boundary of the cube into the basepoint. In [8] the unit interval is replaced by the infinite *fence*,

$$0 \to 1 \leftarrow 2 \to 3 \leftarrow 4 \to 5 \cdots, \tag{1.1}$$

which we denote by F. Thus F is the poset of which the underlying set is the set $\mathbb{N}$ of all non-negative integers, and the partial order $\theta \subset \mathbb{N} \times \mathbb{N}$ is such that

$$(k, l) \in \theta \Leftrightarrow l = k \quad \text{or} \quad |k - l| = 1 \text{ and } k \text{ is even.}$$

For any $k \in \mathbb{N}$, F_k denotes the subobject $\{0, 1, 2, \ldots, k\}$ of F.

In Section 2 we review some basic constructions that will be required in the sequel. In Section 3 we extend the work in [8] by defining analogues in $\mathcal{R}$ of relative homotopy groups for objects of the form $(X, A, *)$, where X is a relational structure, A is a subobject of X and $*$ is a (base) point in A. The groups are denoted by $\sigma_k(X, A, *)$, $k \in \mathbb{N}$. Section 4 comprises some observations and lemmata required to compare, in the case of posets, the newly defined relative group with the classical relative homotopy group. Much of this is adapted from the paper [8] of Larose and Tardif.

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In Section 5 we further analyze analogies of the new theory with homotopy theory, such as the existence of a long exact sequence. We show that in the case that X is a poset, then for every $k \in \mathbb{N}$ the group $\sigma_k(X, A, *)$ is isomorphic to $\pi_k(X, A, *)$, where in the latter case we consider the underlying T_0 spaces instead of posets. This fact implies that every concept which is defined entirely in terms of homotopy groups, can be introduced into $\mathcal{R}$. Thus in Section 6 we discuss weak homotopy equivalences in the relative sense. In particular we prove a generalization of a well-known theorem of Quillen [11, Theorem 6.3] (formulated in this paper as Theorem 6.2) on weak homotopy equivalences. Much further work has been done on posets under this theme initiated by Quillen, see for instance the paper by Björner, Wachs, and Welker [3]. Even the special case of Theorem 6.3, of considering only posets rather than relational structures more generally, is a new contribution in this regard.

2 Some Basic Constructions

We review some basic concepts and some constructions that lead to new relational structures. Some of these can be found in [8]. We include some further insights.

2.1 Connected Sets

Consider any $\mathcal{R}$-object X. For a pair of points $x, y \in X$ we say that $x \sim_c y$ if there is an $\mathcal{R}$-morphism $g\colon F \to X$ (and here F is the fence) such that $x, y \in g(F)$. The relation $\sim_c$ is an equivalence relation on X, and the equivalence classes are called the *connected components* of X or simply the components. If X has only one component, we say that X is *connected*. The set of all components of X will be denoted by $\sigma_0(X)$. We also note that if $f\colon X \to Y$ is an $\mathcal{R}$-morphism and C is a connected subobject of X, then $f(C)$ is a connected subobject of Y. Thus every $\mathcal{R}$-morphism $f\colon X \to Y$ induces a function $f_*\colon \sigma_0(X) \to \sigma_0(Y)$.

Definition 2.1 An $\mathcal{R}$-morphism $f\colon X \to Y$ is said to be a 0-*equivalence* if the function $f_*\colon \sigma_0(X) \to \sigma_0(Y)$ is surjective.

For $k \in \mathbb{N}$ the map $f\colon X \to Y$ is said to be a k-*equivalence* if the following two conditions hold.

(i) $f_*\colon \sigma_0(X) \to \sigma_0(Y)$ is bijective;
(ii) for every $x \in X$, the homomorphism $f_*\colon \sigma_n(X, x) \to \sigma_n(Y, f(x))$ is an isomorphism whenever $1 \leq n \leq k-1$ and an epimorphism if $n = k$.

The map $f\colon X \to Y$ is said to be a *weak equivalence* if it is a k-equivalence for all $k \in \mathbb{N}$.

2.2 Barycentric Subdivision

Consider any relational structure X. By a *chain* in X we mean a subset C of X where C is the image in X of an $\mathcal{R}$-morphism $f\colon T \to X$ where T is a totally ordered set. We recall from [8] that the barycentric subdivision X' of X is defined to be the poset (under set inclusion) of all finite chains in X. Barycentric subdivision can be seen to be a functor from $\mathcal{R}$ to $\mathcal{P}$. We further want to note that in [8], for any $\mathcal{R}$-object X, an $\mathcal{R}$-morphism $p\colon X' \to X$ is shown to exist, such that for every chain τ in X, $p(\tau)$ is an upper bound of τ. Such a map induces an isomorphism of $(\sigma-)$ homotopy groups and is unique if X is antisymmetric. Let us refer to any such map as a *barycentric projection*. We note however that while such a map p is a retraction as a function between the sets, p is not a retraction in $\mathcal{R}$.

Remark 2.2 When constructing barycentric subdivision and projections we find that for a morphism $g\colon X \to Y$, a square of the form

$$\begin{array}{ccc} X' & \xrightarrow{g'} & Y' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow[g]{} & Y \end{array}$$

is not necessarily commutative. However we observe the following.

- If g is injective, then for any choice of α we can find a certain β such that the square above is commutative.

We note that in $\mathcal{R}$ there is a natural way of imposing relational structures on products and hom-sets.

2.3 Product

The *product* of two relational structures (X, θ_X) and (Y, θ_Y) is defined to be the reflexive binary relational structure $(X \times Y, \theta_{X\times Y})$, where $\theta_{X\times Y}$ is the relation on $X \times Y$ described as follows. For $x, x_1 \in X$ and $y, y_1 \in Y$, $((x, y), (x_1, y_1)) \in \theta_{X\times Y}$ if and only if $(x, x_1) \in \theta_X$ and $(y, y_1) \in Y$.

2.4 Structure on Hom-sets

Consider any $\mathcal{R}$-objects X and Y. We denote the set of $\mathcal{R}$-morphisms $X \to Y$ by $\operatorname{Hom}(X, Y)$. Let $H = \operatorname{Hom}(X, Y)$ and let us define a relation θ_H on H as follows. For $f, g \in H$, we let $(f, g) \in \theta_H$ if and only if, whenever $(x, y) \in \theta_X$, then $(fx, gy) \in \theta_Y$. Then θ_H is reflexive and thus $\operatorname{Hom}(X, Y)$ becomes a relational structure.

2.5 Pairs

By a *pair* in $\mathcal{R}$ we shall mean an object X together with a subobject A, and an $\mathcal{R}$-morphism $f\colon X \to Y$ is said to be a map of pairs $f\colon (X, A) \to (Y, B)$ if $f(A) \subseteq B$. A pair (X, A) in $\mathcal{R}$ is called a *pointed* relational structure if A consists of exactly one point. Thus we shall sometimes speak of pointed maps.

2.6 Geometric Realization

Let P be any poset. For any function $h\colon P \to [0,1]$ the *support* $\mathrm{st}(h)$ of h is the subset of P defined by $\mathrm{st}(h) = \{p \in P : h(p) \neq 0\}$(see [8]).

Now consider the set

$$\widehat{P} = \{h \in [0,1]^P : \mathrm{st}(h) \text{ is a finite chain in } P \text{ and } \Sigma_{p\in P}\, h(p) = 1\}.$$

The set $\widehat{P}$ is equipped with the coherent topology (see [12]) and the resulting topological space is called the *geometric realization* of P.

3 The Relative Homotopy Group

Let F be the fence defined in the introduction (1.1). Recall that for any $m \in \mathbb{N}$, by F_m we mean the subposet $\{0,1,2,\dots,m\}$ of F. For the homotopy theory in posets, the posets F_m fulfill the role of the unit interval.

3.1 The Set $\Sigma_k(X, A, *)$

Let X be a relational structure, let A be a subobject of X, and fix a point $*$ in A. We also fix $k \in \mathbb{N}$ and let $K = \{1,2,3,\dots,k\}$. We denote a typical element of F^k by $t = (t_1, t_2, \dots, t_k)$.

For $k = 1$, let $\Sigma_1(X, A, *)$ be the set of all maps $f\colon F \to X$ such that $f(0) \in A$, and for which there exists a least integer $n = N(f)$ such that for every $s \in \mathbb{N}$, $f(n+s) = f(n) = *$.

For any $k \in \mathbb{N}$ with $k \geq 2$, let $\Sigma_k(X, A, *)$ be the set of all $\mathcal{R}$-morphisms $f\colon F^k \to X$ satisfying the following conditions (and here $t \in F^k$):

(i) If $t_k = 0$, then $f(t) \in A$.
(ii) If $t_i = 0$ for some $i \in K\backslash\{k\}$, then $f(t) = *$.
(iii) For each $i \in K$ there exists a natural number $n_i(f)$ such that if $t_i \geq n_i(f)$ for some $i \in K$, then $f(t) = *$.

For a given i, the minimum of the numbers $n_i(f)$ will be denoted by $N_i(f)$ and we let $N(f) = \max\{N_i(f) | i = 1,2,3,\dots,k\}$.

3.2 The Group $\sigma_k(X, A, *)$

The set $\Sigma_k(X, A, *)$, being a set of morphisms between two relational structures, becomes itself a relational structure (see Section 2.6). The set of connected components (which will also be referred to as *homotopy classes*) of $\Sigma_k(X, A, *)$ will be denoted by $\sigma_k(X, A, *)$. For any member g of $\Sigma_k(X, A, *)$, the connected component which contains g will be denoted by $[g]$. Note that in the case $A = *$, the set $\sigma_k(X, A, *)$ coincides with $\sigma_k(X, *)$ as defined in [8, Definition 2.5].

Now let us assume $k \geq 2$. For $f, g \in \Sigma_k(X, A, *)$ and for an even integer N with $N \geq N(f)$ we define

$$(f,g)_N(t_1,t_2,\dots,t_k) = \begin{cases} f(t_1,t_2,\dots,t_k) & \text{if } t_1 \leq N,\\ g(t_1 - N, t_2, \dots, t_k) & \text{otherwise.}\end{cases}$$

We extend the definition of the group operation [8, Definition 2.6] in the case $k \geq 2$ by setting $[f] \cdot [g] = [(f,g)_N]$ for any even integer N with $N \geq N(f)$. Thus, for $k \geq 2$, the same formula used in [8, Definition 2.6] to impose a group structure on $\sigma_k(X, *)$ turns out to give a group structure also on $\sigma_k(X, A, *)$. The unit of the latter group is the connected component of the constant $*$ map in $\Sigma_k(X, A, *)$. Furthermore the relative group $\sigma(-)$ is a functor. It follows that for $k \geq 2$ the group $\sigma_k(X, \{*\}, *)$ is the same as the group $\sigma_k(X, *)$ of [8].

The set of path components of $\Sigma_1(X, A, *)$ is denoted by $\sigma_1(X, A, *)$. We note that $\sigma_1(X, A, *)$ is functorial but is not a group in general.

4 An Approximation Theorem for Maps Into Posets

In this section we introduce some technical terminology and we revisit some key results of [8], formulating the appropriate versions for our purposes. The proofs of such modified results from [8] follows almost verbatim as those of their counterparts in [8], and we omit the proofs. These results are required in Section 5 to prove the exactness of the homotopy sequence of a pointed pair $(X, A, *)$ in $\mathcal{R}$, and in the case of X being a poset, to compare the relative homotopy group $\sigma_k(X, A, *)$ with $\pi_k(X, A, *)$.

4.1 Notation

Just as in [8], for a pointed pair $(X, A, *)$ of posets we define a function

$$\Delta_k \colon \sigma_k(X, A, *) \to \pi_k(X, A, *).$$

For $g \in \Sigma_k(X, A, *)$, let m be any natural number which is not less than $N(g) = \max(S)$ where S is the set $S = \{N_1(g), N_2(g), \dots, N_k(g)\}$. Let $[g]$ be the homotopy class of g in $\sigma_k(X, A, *)$ and let $\overline{g}$ be the restriction of g to F_m^k.

Let $\alpha_m^k \colon \widehat{F_m^k} \to F_m^k$ be the weak equivalence of McCord similar to, for instance, the one in [8, Theorem 3.2].

Let $\phi_m^k \colon \widehat{F_m^k} \to I^k$ be the obvious homeomorphism, similar to the one defined in [8, Definition 3.6], and let ψ_m^k be its inverse.

Now let $\Delta_k[g] = [\overline{g} \circ \alpha_m^k \circ \psi_m^k]$. Then Δ_k is well defined. It is also clear that Δ_k is a homomorphism. In the next section we prove that for each $k \in \mathbb{N}$, Δ_k is an isomorphism.

4.2 Definition

Let $n \geq 2$ be an integer and $k \in \mathbb{N}$.

A subset B of $[0,1]^k$ is an *n-box* if $B = \Pi_{i=1}^k B_i$, where each B_i is an interval of the form $B_i = [0,1] \cap (\frac{a}{n}, \frac{b}{n})$, for odd integers a and b with $a < b$ (*cf.* [8]).

Let P be a poset. A continuous function $f \colon [0,1]^k \to P$ is said to be *n-simple* if for each $p \in P$, the set $\{x \in [0,1]^k : f(x) \leq p\}$ is a union of n-boxes.

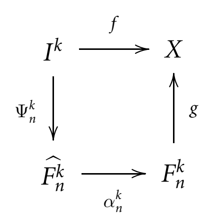
4.3 Notation

Let us fix $k \in \mathbb{N}$ and let $K = \{1, 2, 3, \dots, k\}$. We define the following sets:

$$J_k = \{x \in I^{k+1} : \ x_{k+1} = 1 \text{ or } x_i(1 - x_i) = 0 \text{ for some } i \in K\},$$
$$J_k^* = J_k \cup \{x \in I^{k+1} : x_{k+1} = 0\} = \partial I^{k+1}.$$

In order to prove the exactness of the analogue of the homotopy sequence of a pair of relational structures, we require two auxiliary results which can be derived from work in [8].

Lemma 4.1 (**[8, Lemma 3.15]**) *Let $(P, *)$ be a pointed poset and let $n \geq 4$ be an even integer. A map $f\colon I^k \to P$ is n-simple if and only if there exists a poset map $g\colon F_n^k \to P$ such that $g \circ \alpha_n^k \circ \Psi_n^k = f$,* i.e., *the following diagram commutes.*



Theorem 4.2 (**[8, Theorem 3.17]**) *Let (P, U, V) be a triple in $\mathcal{P}$ with $V \subseteq U \subset X$. Let $f\colon (I^{k+1}, J_k^*, J_k) \to (P, U, V)$ be a continuous map. Then there exists an integer s and an s-simple function $g\colon I^{k+1} \to P$ such that $g \leq f$, $g(J_k^*) \subseteq U$, and $g(J_k) \subseteq V$.*

5 The Homotopy Sequence of a Pair

There is an analogue in $\mathcal{R}$ of the exact homotopy sequence of a pair of spaces. We define the sequence in (5.2), and settle the exactness in Theorem 5.2. We now describe the relevant boundary homomorphism.

5.1 The Boundary Homomorphisms $\partial\colon \sigma_k(X, A, *) \to \sigma_{k-1}(A, *)$

For $k \in \mathbb{N}$ with $k \geq 2$ and for a pair (X, A), we define

(5.1) $$\partial\colon \sigma_k(X, A, *) \to \sigma_{k-1}(A, *)$$

as follows: given any $g \in \Sigma_k(X, A, *)$, let $n = N(g)$ and let $\partial([g]) = [g_0]$, where $g_0\colon F^{k-1} \to X$ is the map $g_0(t_1, t_2, t_3, \dots, t_{k-1}) = g(t_1, t_2, t_3, \dots, t_{k-1}, 0)$. Then $g_0 \in \Sigma_{k-1}(A, *)$, and the induced function $\partial\colon \sigma_k(X, A, *) \to \sigma_{k-1}(A, *)$, is obviously a group homomorphism.

The *injections* $i\colon \sigma_k(A, *) \to \sigma_k(X, *)$ and $j\colon \sigma_k(X, *) \to \sigma_k(X, A, *)$, and the boundary homomorphism (5.1) give rise to a homotopy sequence for a pair in $\mathcal{R}$

such as in (5.2), which is analogous to the homotopy sequence of a pair of pointed topological spaces

$$(5.2)\qquad \cdots \xrightarrow{j} \sigma_{k+1}(X,A,*) \xrightarrow{\partial} \sigma_k(A,*) \xrightarrow{i} \sigma_k(X,*) \xrightarrow{j} \sigma_k(X,A,*).$$

The sequence terminates at the right-hand side in the object $\sigma_1(X,*)$. (Actually we can continue a little further in the right-hand direction, as is the case with the homotopy sequence of a pair of topological spaces, but the relevant objects fail to be groups.)

Theorem 5.1 *Let $k \in \mathbb{N}$ with $k > 1$. Given any $[h_0] \in \sigma_k(X,*)$ for which $j[h_0] = 0 \in \sigma_k(X,A,*)$, then there exists $[h_1] \in \sigma_k(A,*)$ for which $i[h_1] = [h_0]$.*

Proof Suppose that $[h_0] \in \sigma_k(X,*)$ with $j[h_0] = 0 \in \sigma_k(X,A,*)$. The identity $j[h_0] = 0$ implies that there is some $m \in \mathbb{N}$ and a map $H_0 \colon F_m^{k+1} \to X$ such that the following conditions are fulfilled:

- for every $t \in F_m^k$ we have $H_0(t \times 0) = h_0(t)$ and $H_0(t \times m) \in A$;
- if $s \in F_m^{k+1}$ and $s_k = 0$, then $H_0(s) \in A$;
- if $s \in F_m^{k+1}$ and $(s_k - m)(s_u - m)s_u = 0$ for some $u < k$, then $H_0(s) = *$.

We define a map $g \colon I^k \to I^{k+1}$ as follows (and here we consider x to denote a typical element of I^{k-1}, and $s \in I$):

$$g(x,s) = \begin{cases} (x,1,2s) & \text{if } s \le \frac{1}{2}, \\ (x,2-2s,1) & \text{otherwise.} \end{cases}$$

Then g is well defined and by the pasting lemma, g is continuous. Let

$$W = \{y \in I^{k+1} : y_k(y_k - 1)y_{k+1}(y_{k+1} - 1) = 0\}.$$

Let us define a map $f_0 \colon W \to W$ by the following formula (and again we consider x to denote a typical element of I^{k-1} while $s, t \in I$):

$$f_0(x,s,t) = \begin{cases} g(x,t) & \text{if } s = 1, \\ (x,0,s) & \text{if } t = 1, \\ (x,0,0) & \text{if } s = 0, \\ (x,s,0) & \text{if } t = 0. \end{cases}$$

Then f_0 (is well-defined and continuous, and) can be extended to a map $f_1 \colon J_k^* \to J_k^*$ which is such that $f_1(J_k^* \setminus W) \subseteq J_k^* \setminus W$. Furthermore, f_1 admits an extension $f_2 \colon I^{k+1} \to I^{k+1}$. Let $f_3 = \alpha_m^{k+1} \circ \psi_m^{k+1} \circ f_2$.

By Theorem 4.2 there exists $M \in \mathbb{N}$ and a map $H_1 \colon I^{k+1} \to F_m^{k+1}$ such that H_1 is M-simple, $H_1 \le f_3$, and $H_1(I^k \times \{0,1\}) \subseteq V \subseteq H_0^{-1}(*)$ for some open subset V of F_m^{k+1}.

By Lemma 4.1 there exists a map $H_2 \colon F_M^{k+1} \to F_m^{k+1}$ such that the following diagram commutes.

$$\begin{array}{ccc} I^{k+1} & \xrightarrow{H_1} & F_m^{k+1} \\ \Big\downarrow{\scriptstyle \Psi_M^{k+1}} & & \Big\uparrow{\scriptstyle H_2} \\ \widehat{F_M^{k+1}} & \xrightarrow{\alpha_M^{k+1}} & F_M^{k+1} \end{array}$$

We choose $h_1 \colon F_M^k \to A$ to be the map $h_1 \colon t \mapsto H_0 \circ H_2(t \times M)$. ■

Theorem 5.2 *The homotopy sequence of the pair* (X, A) *of objects in* $\mathcal{R}$ *defined in* (5.2) *is exact.*

Proof It is not hard to see that the sequence is differential. By Theorem 5.1 it follows that the sequence is exact at the objects $\sigma_k(X, *)$. Exactness at the other objects follows relatively easily. ■

We noted in Remark 2.2 that for a pair of relational structures (X, A) there is the barycentric subdivision (see [8]) which gives rise to a pair (X', A') of posets, and one can choose a barycentric projection map $\beta : X' \to X$ which is consistent with the pairs, *i.e.*, it actually yields a map $\beta \colon (X', A') \to (X, A)$. This idea and notation is used in the second part of the following theorem.

Theorem 5.3 (i) *For a pointed pair of posets,* $(X, A, *)$, *the homomorphism* Δ_k : $\sigma_k(X, A, *) \to \pi_k(X, A, *)$ *is an isomorphism.*

(ii) *For a pointed pair of relational structures* $(X, A, *)$ *and a barycentric projection map* $\beta \colon (X', A', *) \to (X, A, *)$, *the induced homomorphisms* $\sigma_k(X', A', *) \to \sigma_k(X, A, *)$ *are isomorphisms.*

Proof (i) The homotopy sequence of a pair of spaces is exact and by Theorem 5.2 the σ-homotopy sequence of a pair of posets is exact. Furthermore we note that the morphisms $\Delta_k(-)$ are in fact natural transformations in both the absolute and the relative cases. We thus consider the ladder-shaped diagram formed by the exact $(\sigma-)$ homotopy sequence of the poset pair $(X, A, *)$, the exact homotopy sequences of the pair of topological spaces $(X, A, *)$ and the homomorphisms Δ_k. The assertion of the theorem follows by application of the 5-lemma since, as proved in [8], the morphisms $\Delta_k \colon \sigma_k(A) \to \pi_k(A)$ and $\Delta_k \colon \sigma_k(X) \to \pi_k(X)$ are isomorphisms.

(ii) This follows from the 5-lemma applied to the ladder-shaped diagram formed by the exact (σ)-homotopy sequences of the pairs $(X', A', *)$ and $(X, A, *)$ and the homomorphisms induced by β, using the fact that the induced maps $X' \to X$ and $A' \to A$ are known (by [8]) to be weak equivalences. ■

6 Weak Equivalences and Quasifibrations

Now that we have the analogues of absolute and relative homotopy groups in the category $\mathcal{R}$, we can introduce into $\mathcal{R}$ certain homotopical terms defined in terms of homotopy groups such as weak equivalence of a map of pointed objects and weak equivalence of a map of pointed pairs.

Definition 6.1 (**[9, 13]**) A map $p\colon (X,A) \to (Y,B)$ of pairs in $\mathcal{R}$ is a 0-*equivalence* if the first condition below holds. If n is a positive integer, then f is said to be an n-*equivalence* if both conditions hold. (In (i) below, the functions between path components are the functions induced by the relevant inclusion maps.):

(i) $\operatorname{Im}[\sigma_0(A) \to \sigma_0(X)] = p_*^{-1} \operatorname{Im}[\sigma_0(B) \to \sigma_0(Y)]$,

(ii) For every $a \in A$, and with $b = p(a)$, the function $p_*\colon \sigma_r(X,A,a) \to \sigma_r(Y,B,b)$ is bijective whenever $r < n$ and surjective for $r = n$.

The map p is said to be a *weak equivalence* if it is a n-equivalence for all $n > 0$. A map $p\colon X \to Y$ is said to be a *quasifibration* if for every $y \in Y$ and $G = p^{-1}(y)$, the induced map of pairs $(X,G) \to (Y,y)$ is a weak equivalence. A relational structure C is said to be *weakly contractible* if for any $x \in C$ the inclusion map $\{x\} \to C$ is a weak equivalence.

Quasifibrations were used extensively in topology since the paper [5] of Dold and Thom. Especially the theorem [5, Satz 2.2] has found numerous applications. The latter theorem gives a sufficient condition for a map $E \to B$ (of topological spaces) to be a quasifibration in terms of the behaviour of the map with respect to members of a basis-like collection of subsets of B. J. P. May [9] gave a different approach to quasifibrations, in terms of weak equivalences between pairs of spaces. The main theorem of [13] can be regarded as a generalization of [5, Satz 2.2] and the main theorem of [9].

An important theorem of Quillen [11, Proposition 1.6] is the following, which is often used to prove (weak) equivalences of posets, especially subgroup posets of finite groups. We formulate it in the context of this paper.

Theorem 6.2 *Let $f\colon X \to Y$ be a map of posets. Suppose that for every $y \in Y$ the subobject $\{x \in X : f(x) \le y\}$ is weakly contractible. Then f is a weak equivalence.*

We now present a generalization of Theorem 6.2. In the proof of Theorem 6.3 we use ideas and terminology from [13]. We recall from Section 2 that for an object X in $\mathcal{R}$, a chain in X is the image in X of a morphism $f\colon T \to X$ where T is a totally ordered finite set.

Theorem 6.3 *Let $p\colon X \to Y$ be a map in $\mathcal{R}$ with Y connected. Suppose that the following condition holds:*

(C1) *Whenever C is a finite chain in Y and $D \subseteq C$, then the inclusion map $p^{-1}(D) \to p^{-1}(C)$ is a weak equivalence.*

Then for each chain A in Y, the map $(X, p^{-1}(A)) \to (Y,A)$ is a weak equivalence.

Proof Let $p^{-1}(A) = B$. We note that we can choose weak equivalences $\alpha\colon X' \to X$ and $\beta\colon Y' \to Y$ in such a way that $\beta(B') = B$ and $\alpha(A') = A$, and such that the following square is homotopy commutative. (Recall that we denote the barycentric subdivision of a structure Z by Z'.)

$$\begin{array}{ccc} X' & \xrightarrow{p'\ (=q)} & Y' \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ X & \xrightarrow{p} & Y \end{array}$$

Moreover, then the induced maps $A' \to A$ and $B' \to B$ are weak equivalences. The barycentric subdivision is such that if condition (C1) holds for p, then the following condition (C2) holds for p' (and for convenience we write $q = p'$):

(C2) Whenever $C \in \mathcal{U}(Y')$ (see the notation specified in Section 1.1) and $D \subseteq C$, then the inclusion map $q^{-1}(D) \to q^{-1}(C)$ is a weak equivalence.

The morphism q can, of course, be considered a morphism of topological spaces. By [13] it follows that for every $V \in \mathcal{U}(Y')$, the morphism $(X', q^{-1}(V)) \to (Y', V)$ is a weak equivalence. Since the diagram above is homotopy commutative and since we have weak equivalences $X' \to X$, $B' \to B$, $Y' \to Y$, and $A' \to A$, it follows that $(X, B) \to (Y, A)$ is a weak equivalence. ∎

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