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ORIGINAL PAPER

Some Meta-Cayley Graphs on Dihedral Groups

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Abstract

In this paper, we define meta-Cayley graphs on dihedral groups. We fully determine the automorphism groups of the constructed graphs in question. Further, we prove that some of the graphs that we have constructed do not admit subgroups which act regularly on their vertex set; thus proving that they cannot be represented as Cayley graphs on groups.

Keywords Vertex-transitive graphs · Cayley graphs · Groupoid graphs · Non-Cayley graphs · Meta-Cayley graphs

Mathematics Subject Classification 05E18 · 05C25

1 Introduction

Through computational observations, it has been conjectured by Ivanov and Praeger [6] that

$$\lim_{n\to\infty} \frac{\mathrm{cay(n)}}{\mathrm{vtr(n)}} = 1,$$

where $\mathrm{vtr}(n)$ and $\mathrm{cay}(n)$ denote the number of isomorphism types of vertex-transitive and Cayley graphs on groups, respectively, with order at most $n \geq 1$. In other words, it is conjectured that the majority of vertex-transitive graphs are Cayley on groups. That is, the majority of vertex-transitive graphs can be represented by a group and a Cayley set. Much earlier, in [8] Marušič asked for which positive integers n does there exist a vertex-transitive graph on n vertices which is not Cayley on groups. This means

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that there are indications that vertex-transitive graphs that are non-Cayley on groups are a rarity in occurrence. The determination of vertex-transitive graphs that are non-Cayley on groups has thus raised a lot of attention. There has been notable success in the construction of such graphs; see, for instance, [1,2,5,7,9,10,16,18]. Henceforth, we will refer to vertex-transitive graphs which are non-Cayley on groups by the acronym VTNCGs introduced by Watkins [18].

A graph $\Gamma = (V, E)$ is a set of vertices V and an irreflexive and symmetric relation E on V whose elements are called the edges of the graph and denoted $[x, y]$, if $e = [x, y] \in E$. For $e = [x, y] \in E$, it is said that vertices x and y are adjacent. A sequence of vertices $x_0, x_1, \ldots, x_k$ such that each x_i and x_{i+1} are adjacent and all vertices besides x_k are distinct, is called a path. If $x_0 = x_k$ then the path is called a cycle. An automorphism g of a graph $\Gamma = (V, E)$ is a permutation of the vertex set V such that $e^g \in E$ whenever $e \in E$. The set of automorphisms of a graph Γ form a group under composition and is denoted $\mathrm{Aut}\Gamma$. A graph Γ is vertex-transitive if $\mathrm{Aut}\Gamma$ acts transitively on V. Let G be a group and S a subset of G such that $1_G \notin S$ and $s^{-1} \in S$ whenever $s \in S$. Such a set S is called a Cayley set. Then the Cayley graph on the group G with respect to S, $\mathrm{Cay}(G, S)$, is the graph with the vertex set G such that $[x, y]$ is an edge if and only if $y = xs$ for some s in S. For a fixed element $g \in G$, it is easy to see that the map $\lambda_g : G \longrightarrow G$ defined by $\lambda_g(a) = ga$ for any $a \in G$ defines an automorphism of the graph $\mathrm{Cay}(G, S)$. Moreover, it is easy to see that the set of left translations $\Lambda_G := \{\lambda_g : g \in G\}$ acts transitively on $\mathrm{Cay}(G, S)$ so that Cayley graphs on groups are vertex-transitive.

A groupoid $(A, *)$ is a set A endowed with a binary operation $* : A \times A \to A$. It is often written A instead of $(A, *)$. In the finite case, a left[right] loop is a groupoid which admits left[right] cancellation and has an identity. If it is both left and right cancellative and has an identity, then it is called a loop. The notion of Cayley graphs which are defined on groups has been generalised to groupoids in [12] as follows. A subset S of a groupoid A is said to be Cayley if $a \notin aS$ for any $a \in A$ and $a \in (as)S$ for any $a \in A$ and $s \in S$. The corresponding Cayley graph on the groupoid $\Gamma = \mathrm{Cay}(A, S)$ has vertex set $V(\Gamma) = A$ and edge set $E(\Gamma) = \{[a, as] : a \in A, s \in S\}$.

A subset S of a groupoid A is said to be quasi-associative if for every $x, y \in A$ we have that

$$x(yS) = (xy)S.$$

The concept of quasi-associative subsets was introduced by Gauyacq [4] who termed them right associative. As elsewhere [12–15], we will continue to call them quasi-associative. Cayley graphs on loops with quasi-associative Cayley subsets are termed quasi-Cayley. In this general setting, it is easy to see that left translations are automorphisms of $\mathrm{Cay}(A, S)$ whenever A is a left loop and S is quasi-associative. Moreover, just as in groups, the set $\Lambda_A := \{\lambda_a : a \in A\}$ acts regularly on A; except that the set is not necessarily a group under composition.

In [13], it has been shown that every vertex-transitive graph is isomorphic to a Cayley graph on a left loop with respect to a quasi-associative Cayley set. In view of this, Marušič's question translates to finding Cayley graphs on left loops with respect to quasi-associative Cayley sets which cannot be represented as Cayley graphs on groups.

This paper presents vertex-transitive graphs of order $n = 4m$ which cannot be represented on groups. We construct loops by defining a semi-direct product on groups A and A' with the provision that the twisting map $f : A \to \mathrm{Aut}A'$ is not necessarily a group homomorphism. Instead, it is only required that the map f satisfies a rather weak condition to obtain loops. In addition, the crux of the matter is to determine subsets of the product that are quasi-associative and Cayley to construct vertex-transitive graphs. Such graphs have been called meta-Cayley graphs [15]. In order to prove that graphs of this class are non-Cayley on groups, we first completely determine their automorphism groups. We will then apply Sabidussi's theorem [17]. Sabidussi's theorem states that a graph is a Cayley graph on a group if and only if the automorphism group contains a subgroup which acts regularly on the vertex set.

The rest of the paper is organised as follows. In Sect. 2, we present loops as semi-direct products of groups on dihedral groups and outline our approach in finding the class of VTNCGs. Further we construct meta-Cayley graphs. In Sect. 3, we fully determine the automorphism groups of our constructed graphs and apply Sabidussi's theorem.

2 Loops and Their Corresponding Meta-Cayley Graphs

For groups A and A' and mapping $f : A \to \mathrm{Aut}A'$, we denote a semi-direct product of A and A' as $A \times_f A$, where the elements of $A \times_f A$ are of the form $(a, f_a(a'))$ with $a \in A$ and $a' \in A'$. As alluded to in the introduction, a semi-direct product on two groups whose twisting map satisfies a weak condition to obtain a loop and not a group were introduced in [15] in the following way.

Proposition 1 [15] *Let A and A' be groups. Let f be a mapping $f : A \to \mathrm{Aut}A'$ such that $f(e)$ is the identity map on A', where e is the identity element of A. Denote $f(a)$ as f_a and define Q by $Q = A \times_f A'$. Then Q is a loop.*

Again, as was also alluded to earlier, determining quasi-associative Cayley subsets is at the crux of the matter. It has been observed in [15] that sets satisfying the following are quasi-associative Cayley subsets.

Proposition 2 [15] *Let A and A' be groups and Q be as in Proposition 1. For each $x \in A$, let L_x be a (possibly empty) subset of A' such that*

(a) *$e' \notin L_e$ where e and e' are the identity elements of A and A' respectively;*
(b) *$L_{x^{-1}} = (f_x^{-1}[L_x])^{-1}$ for any $x \in A$;*
(c) *$f_a f_b[L_x] = f_{ab}[L_x]$ for any $a, b \in A$.*

Let U be the subset of Q defined by

$$U := \bigcup_{x \in A} x \times L_x.$$

Then U is a quasi-associative Cayley subset of Q.

With Q and U as in Propositions 1 and 2 above, the resultant graph $\mathrm{Cay}(Q, U)$ is then necessarily vertex-transitive. Formally, the graphs we consider are defined in the following way.

Definition 1 Let Q be as in Proposition 1 and U be as in Proposition 2. We call U a *meta-Cayley* subset of Q and the graph $\mathrm{Cay}(Q, U)$ is called a *meta-Cayley* graph.

In this paper, we consider the case $A = \mathbb{Z}_2$ and $A' = D_n$. To facilitate our discussion, we will represent the dihedral group D_n as $\mathbb{Z}_2 \times_g \mathbb{Z}_n$ where $g : \mathbb{Z}_2 \to \mathrm{Aut}\mathbb{Z}_n$ defined by $g_x(y) = (-1)^x y$. Hence as is customary, elements of the form $(0, y)$ represent rotations and elements $(1, y)$ represent reflections. The binary operation on the dihedral group is defined as

$$(x, y) \oplus (x', y') = \left(x + x', y + g_x(y')\right).$$

To define our loops, we have to deal with two twisting maps: g to define dihedral groups on the set $\mathbb{Z}_2 \times \mathbb{Z}_n$ and f to define loops. Now, let r be an integer such that $(n, r) = 1$. We define f by $f_x(y, z) = (y, r^x z)$. For such an r, it is a classical result that f_x is an automorphism on D_n. Note that any automorphism will respect the first co-ordinate since any automorphism on D_n preserves both the set of reflections and rotations (see Miller [11]).

The groupoid is therefore fully defined as $Q(n, r) = \mathbb{Z}_2 \times_f (\mathbb{Z}_2 \times_g \mathbb{Z}_n)$ with $(n, r) = 1$ and the binary operation by

$$\begin{aligned}
(x, (y, z)) \oplus \left(x', (y', z')\right) &= \left(x + x', (y, z) \oplus f_x\left(y', z'\right)\right) \\
&= \left(x + x', (y, z) \oplus \left(y', r^x z'\right)\right) \\
&= \left(x + x', \left(y + y', z + g_y\left(r^x z'\right)\right)\right) \\
&= \left(x + x', \left(y + y', z + (-1)^y r^x z'\right)\right)
\end{aligned}$$

with addition modulo 2 in the first and second co-ordinates and modulo n in the third co-ordinate. Henceforth we will not explicitly state this. For brevity, we denote $(x, (y, z))$ as (x, y, z) and $(x, y, z) \oplus (x', y', z')$ as $(x, y, z)(x', y', z')$.

Proposition 3 *Let* $(n, r) = 1$ *and* $Q(n, r) = \mathbb{Z}_2 \times_f (\mathbb{Z}_2 \times_g \mathbb{Z}_n)$ *and define a binary operation on* $Q(n, r)$ *by* $(x, y, z)(x', y', z') = (x + x', y + y', z + (-1)^y r^x (z'))$. *Then* $Q(n, r)$ *is a loop with* $(0, 0, 0)$ *as the identity element.*

Proof Immediate.

Even with this weak twisting map, it may happen that $Q(n, r)$ is a group. Let us immediately deal with this issue by determining under which conditions is $Q(n, r)$ a group.

Proposition 4 *Let* n, r *be integers and* $Q(n, r) = \mathbb{Z}_2 \times_f (\mathbb{Z}_2 \times_g \mathbb{Z}_n)$. *Define a binary operation on* $Q(n, r)$ *by* $(x, y, z)(x', y', z') = (x + x', y + y', z + (-1)^y r^x (z'))$. *Then* $Q(n, r)$ *is a group if and only if* $r^2 \equiv 1 \pmod{n}$.

Proof In view of Proposition 3, it is enough to show associativity. Now, on one hand $((x_0, x_1, x_2)(y_0, y_1, y_2))(z_0, z_1, z_2)$ simplifies to $(x_0 + y_0 + z_0, x_1 + y_1 + z_1, x_2 + (-1)^{x_1} r^{x_0} y_2 + (-1)^{x_1+y_1} r^{x_0+y_0} z_2)$ and on the other $(x_0, x_1, x_2)((y_0, y_1, y_2)(z_0, z_1, z_2))$ simplifies to $(x_0 + y_0 + z_0, x_1 + y_1 + z_1, x_2 + (-1)^{x_1} r^{x_0}(y_2 + (-1)^{y_1} r^{y_0} z_2))$.

So for associativity to hold, it is required that we have

$$\begin{aligned}
&x_2 + (-1)^{x_1} r^{x_0}(y_2) + (-1)^{x_1+y_1} r^{x_0+y_0} z_2 = x_2 + (-1)^{x_1} r^{x_0} \left(y_2 + (-1)^{y_1} r^{y_0} z_2\right) \\
&\quad \Leftrightarrow (-1)^{x_1} r^{x_0} y_2 + (-1)^{x_1+y_1} r^{x_0+y_0} z_2 = (-1)^{x_1} r^{x_0} y_2 + (-1)^{x_1} r^{x_0} (-1)^{y_1} r^{y_0} z_2 \\
&\quad \Leftrightarrow (-1)^{x_1+y_1} r^{x_0+y_0} z_2 = (-1)^{x_1} r^{x_0} (-1)^{y_1} r^{y_0} z_2.
\end{aligned}$$

Now, $(-1)^{x_1+y_1} = (-1)^{x_1}(-1)^{y_1}$ for all possible values of x_1 and y_1. If either $x_0 = 0$ or $y_0 = 0$ then it is clear that $r^{x_0+y_0} = r^{x_0} r^{y_0}$. If $x_0 = y_0 = 1$ however, then for the equation to hold we require that $r^{1+1} = r^1 r^1$, which is true if and only if $r^2 \equiv 1 \pmod{n}$. Therefore, $Q(n, r)$ is a group if and only if $r^2 \equiv 1 \pmod{n}$.

In view of Proposition 4 we do not consider graphs where $r^2 \equiv 1 \pmod{n}$. We will therefore focus on graphs where $r^2 \equiv -1 \pmod{n}$, as this would also allow us to find a general form of quasi-associative Cayley sets. We also note that $r^2 \equiv -1 \pmod{n}$ implies $(n, r) = 1$, a fact we use implicitly going forward. As mentioned, central to the construction of the graphs is the identification of these quasi-associative Cayley sets. In the loops $Q(n, r)$ we consider, they are in the following form.

Lemma 1 *Let $r \in \mathbb{Z}_n$ such that $r^2 \equiv -1 \pmod{n}$. Let $Q(n, r) = \mathbb{Z}_2 \times_f (\mathbb{Z}_2 \times_g \mathbb{Z}_n)$ with binary operation defined by $(x, y, z)(x', y', z') = (x+x', y+y', z+(-1)^y r^x(z'))$ and let U be a subset of $Q(n, r)$. Then U is a meta-Cayley subset of $Q(n, r)$ if and only if*

(i) $(0, 0, 0) \notin U$,
(ii) $(0, 0, i) \in U$ *implies* $(0, 0, -i) \in U$,
(iii) $(0, 1, i) \in U$ *implies* $(0, 1, -i) \in U$,
(iv) $(1, 0, i) \in U$ *implies* $(1, 0, -i), (1, 0, ri), (1, 0, -ri) \in U$, *and*
(v) $(1, 1, i) \in U$ *implies* $(1, 1, -i), (1, 1, ri), (1, 1, -ri) \in U$.

Proof We require sets L_0 and L_1 that satisfy the conditions (a), (b) and (c) of Proposition 2.

(a) By (i), $(0, 0, 0) \notin U$.
(b) $L_{0^{-1}} = f_0^{-1}[L_0^{-1}]$ which implies that $L_0 = L_0^{-1}$. Now $(0, i)^{-1} = (0, -i)$ and $(1, i)^{-1} = (1, i)$ in $\mathbb{Z}_2 \times_g \mathbb{Z}_n$. Therefore $(0, i) \in L_0$ implies $(0, -i) \in L_0$, which means that $(0, 0, i) \in U$ implies $(0, 0, -i) \in U$.
$L_{1^{-1}} = f_1^{-1}[L_1]^{-1}$ which implies that $L_1 = f_1^{-1}[L_1]^{-1}$. Now $f_1^{-1}((0, i)^{-1}) = f_1^{-1}((0, -i)) = (0, ri)$. Therefore $(0, i) \in L_1$ implies $(0, ri), (0, -i), (0, -ri) \in L_1$. Similarly, $(1, i) \in L_1$ implies $(1, -ri), (1, -i), (1, ri) \in L_1$. This means that $(1, 0, i) \in U$ implies $(1, 0, ri), (1, 0, -i), (1, 0, -ri) \in U$ and $(1, 1, i) \in U$ implies $(1, 1, ri), (1, 1, -i), (1, 1, -ri) \in U$.
(c) Since f_0 is identity map we need only consider when $a = b = 1$. In which case, we have $f_1 f_1[L_x] = f_0[L_x]$. Therefore $(j, i) \in L_x$ implies that $f_1 f_1(j, i) =$

$(j, -i) \in L_x$. This means that $(j, i) \in U$ implies $(j, -i) \in U$. In particular, $(0, 1, i) \in U$ implies $(0, 1, -i) \in U$.

Now that we have identified possibilities of meta-Cayley subsets, let us discuss the nature of the adjacency that may ensue on these graphs. To facilitate the discussion we first partition $V(\mathrm{Cay}(Q(n, r), U))$ into four natural sets as follows. Define

$$V_{00} := \{(0, 0, i) : i \in \mathbb{Z}_n\}; \tag{1}$$
$$V_{10} := \{(1, 0, i) : i \in \mathbb{Z}_n\}; \tag{2}$$
$$V_{11} := \{(1, 1, i) : i \in \mathbb{Z}_n\}; \tag{3}$$
$$V_{01} := \{(0, 1, i) : i \in \mathbb{Z}_n\}. \tag{4}$$

Now, we note that we have the following admissible adjacencies.

Lemma 2 *Let $Q(n, r)$ be as in Lemma 1 and let U be a meta-Cayley subset of $Q(n, r)$. Let $V(\mathrm{Cay}(Q(n, r), U))$ be partitioned as in* (1)–(4). *In* $\mathrm{Cay}(Q(n, r), U)$

(a) *U contains element(s) of the form* $(1, 0, i)$ *if and only if*
   (i) *every vertex in V_{00} is adjacent to some vertex in V_{10}, and*
   (ii) *every vertex in V_{01} is adjacent to some vertex in V_{11}.*
(b) *U contains element(s) of the form* $(1, 1, i)$ *if and only if*
   (i) *every vertex in V_{00} is adjacent to some vertex in V_{11}, and*
   (ii) *every vertex in V_{01} is adjacent to some vertex in V_{10}.*
(c) *U contains element(s) of the form* $(0, 1, i)$ *if and only if*
   (i) *every vertex in V_{00} is adjacent to some vertex in V_{01}, and*
   (ii) *every vertex in V_{10} is adjacent to some vertex in V_{11}.*
(d) *U contains element(s) of the form* $(0, 0, i)$ *if and only if every vertex is adjacent to some vertex in the same set in the partition.*

Proof Immediate.

It is not difficult to see that for disconnected vertex-transitive graphs, each component is necessarily vertex-transitive and they are isomorphic to each other. Further, for a disconnected VTNCG, each component is itself a connected VTNCG. For this reason, in the pursuit of VTNCGs, we generally only consider the connected case. We carefully choose U so that $\mathrm{Cay}(Q(n, r), U)$ is connected. In view of Lemmas 1 and 2, we have the following.

Lemma 3 *Let* $U = \{(1, 0, j), (1, 0, -j), (1, 0, rj), (1, 0, -rj), (0, 1, i), (0, 1, -i)\}$ *be a meta-Cayley subset of $Q(n, r)$ where i and j are fixed elements in $\mathbb{Z}_n$, $Q(n, r)$ is as in Lemma 1, n is odd and* $(i, n) = 1$. *Then* $\mathrm{Cay}(Q(n, r), U)$ *is connected.*

Proof By Lemma 2, we have that every vertex in V_{00} is adjacent to some vertex in V_{10}, every vertex in V_{01} is adjacent to some vertex in V_{11}, every vertex in V_{00} is adjacent

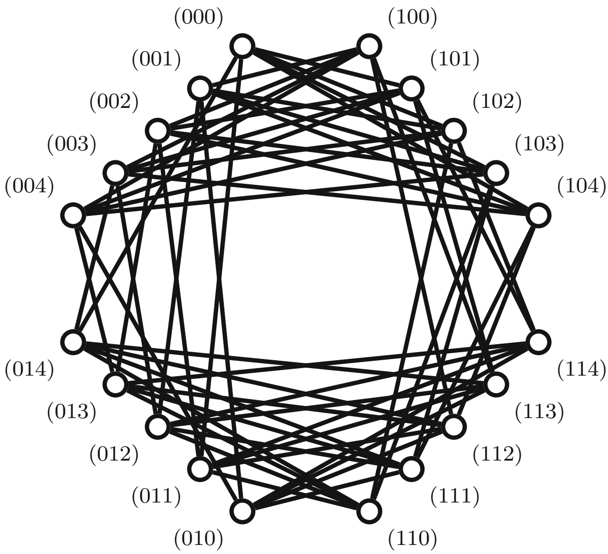

Fig. 1 $\mathrm{Cay}(Q(5,2),U)$

to some vertex in V_{01} and every vertex in V_{10} is adjacent to some vertex in V_{11}. Since $(i,n)=1$, we have that set of edges between elements of V_{00} and elements of V_{01} form a cycle. Since $(i,n)=1$ and $(r,n)=1$, we have that set of edges between elements of V_{10} and elements of V_{11} form a cycle. Therefore, there exists a connecting path between any two vertices of $\mathrm{Cay}(Q(n,r),U)$.

Proposition 5 *Let U be as in Lemma 3 and $Q(n,r)$ be as in Lemma 1 with $i=j$. Let $U'=\{(1,0,1),(1,0,-1),(1,0,r),(1,0,-r),(0,1,1),(0,1,-1)\}$. Then* $\mathrm{Cay}(Q(n,r),U)\cong\mathrm{Cay}(Q(n,r),U')$.

Proof It can easily be checked that the mapping $\gamma:(x,y,z)\to(x,y,zi)$ is an isomorphism from $\mathrm{Cay}(Q(n,r),U')$ to $\mathrm{Cay}(Q(n,r),U)$.

In view of the above discussion, we consider graphs $\mathrm{Cay}(Q(n,r),U)$ where

$$U=\{(1,0,1),(1,0,-1),(1,0,r),(1,0,-r),(0,1,1),(0,1,-1)\},$$

n is odd and $r^2\equiv-1 \pmod n$, recalling that consequently $(n,r)=1$ as well. Note that U has four elements of the form $(1,0,i)$ and two of the form $(0,1,i)$. Figure 1 is the graph $\mathrm{Cay}(Q(5,2),U)$.

3 The Automorphism Groups of $\mathrm{Cay}(Q(n,r),U)$

Having defined the meta-Cayley graphs $\mathrm{Cay}(Q(n,r),U)$, in this section we fully determine the automorphism groups of the graphs. At the end of the section, we will show that the automorphism groups of the graphs do not admit any subgroup which acts regularly on vertex sets. For brevity, we first define notation for our graphs.

Definition 2 Let $Q(n,r) = \mathbb{Z}_2 \times_f (\mathbb{Z}_2 \times_g \mathbb{Z}_n)$ be a loop with binary operation $(x,y,z)(x',y',z') = (x+x', y+y', z+(-1)^y r^x(z'))$ where n, r are integers such that n is odd and $r^2 \equiv -1 \pmod{n}$. Let $U = \{(1,0,1), (1,0,-1), (1,0,r), (1,0,-r), (0,1,1), (0,1,-1)\}$. We denote the vertex-transitive graph $\mathrm{Cay}(Q(n,r), U)$ as $\Gamma(n,r)$.

To facilitate the determination of the automorphism groups, we partition the edge set E in the following way. This partition respects the vertex set partition from the previous section.

$$L := \{[u,v] \in E : u \in V_{00}, v \in V_{01}\}; \tag{5}$$
$$R := \{[u,v] \in E : u \in V_{10}, v \in V_{11}\}; \tag{6}$$
$$M_1 := \{[u,v] \in E : u \in V_{00} \textit{ and } v \in V_{10}\}; \tag{7}$$
$$M_2 := \{[u,v] \in E : u \in V_{11} \textit{ and } v \in V_{01}\}. \tag{8}$$

This partition will assist us to determine the orbits of the automorphism groups of the graphs. Further to this, we also look at the set

$$M := M_1 \cup M_2. \tag{9}$$

Lemma 4 *Let L, R and M be as defined in (5)–(9). If $\gamma \in \mathrm{Aut}\,\Gamma(n,r)$ fixes any of L,R or M set-wise, then it fixes all three or fixes M and interchanges L and R.*

Proof Suppose γ fixes R set-wise. Any edge in L is incident to four edges in M and one edge in L at either end vertex. Let γ map some edge in L onto some edge in M. Then, of the five edges incident at one end, two have to be mapped to edges in R, since every edge in M is incident to two edges in R. This contradicts that R is fixed. Therefore, if R is fixed, then L and M are also fixed. A similar argument holds if L is fixed.

Suppose γ fixes M set-wise. Now, the edges in L and edges in R induce $2n$-cycles. Thus, if L and R are not fixed or interchanged, then we will have some edge in L being incident to some edge in R, which does not occur. This completes the proof.

Our first consideration is finding $(\mathrm{Aut}\,\Gamma)_{L,R,M}$, the set stabiliser of L, R and M. As mentioned in the proof of Lemma 4, the edges of L and R form $2n$-cycles. Since the automorphism group of a cycle consists only of rotations and reflections, we define the following:

$$\rho : (x,y,z) \to (x, y+1, z+1); \tag{10}$$
$$\delta : (x,y,z) \to (x, y, -z). \tag{11}$$

Note that ρ represents rotations on the $2n-$cycle of L with $\rho^{2n} = 1$ and δ represents a reflection on L with $\delta^2 = 1$. It is easily checked that $\rho, \delta \in \mathrm{Aut}\,\Gamma(n,r)$. We now determine the set-stabilizer of L,R,M .

Lemma 5 *Let L, R and M be as defined in* (5)–(9)*. Let* $(\mathrm{Aut}\Gamma(n,r))_{L,R,M}$ *be the set stabiliser of* L, R, M *in* $\mathrm{Aut}\Gamma(n,r)$*, and* ρ, δ *be the maps defined as in* (10) *and* (11)*. Then* $(\mathrm{Aut}\Gamma(n,r))_{L,R,M} = \langle\rho,\delta\rangle$.

Proof Any automorphism in $(\mathrm{Aut}\Gamma(n,r))_{L,R,M}$ must preserve the $2n$-cycles induced by the edges in L and R. All the automorphisms which preserve the induced $2n$-cycles are contained in $\langle\rho,\delta\rangle$. Therefore $(\mathrm{Aut}\Gamma(n,r))_{L,R,M} = \langle\rho,\delta\rangle$.

Let us now consider automorphisms in $(\mathrm{Aut}\Gamma(n,r))_M$ which interchange L and R. By Lemma 4 we will then have entirely determined $(\mathrm{Aut}\Gamma(n,r))_M$. We define another mapping

$$\alpha : (x, y, z) \to (x+1, y, rz). \tag{12}$$

It can easily be checked that $\alpha \in (\mathrm{Aut}\Gamma(n,r))_M$. We note also that $\alpha^4 = 1$ and $\alpha^2 = \delta$. The following Lemma will be important in determining $(\mathrm{Aut}\Gamma(n,r))_M$.

Lemma 6 *Let L, R and M be as defined in* (5)–(9)*. Let* $\gamma \in (\mathrm{Aut}\Gamma(n,r))_M$ *that interchanges L and R. If* $\gamma(0,y,z) = (1,y',z')$ *then* $\gamma(1,y,z) = (0,y',z')$.

Proof If $\gamma(0,y,z) = (1,y',z')$ then $\gamma(0,y+1,z+1) = (1,y'+1,z'+r)$ and $\gamma(0,y+1,z-1) = (1,y'+1,z'-r)$, or $\gamma(0,y+1,z+1) = (1,y'+1,z'-r)$ and $\gamma(0,y+1,z-1) = (1,y'+1,z'+r)$. Since γ preserves the cycles formed by the edge sets L and R, we must consequently have that $\gamma(0,y+i,z+i) = (1,y'+i,z'+ir)$ or $\gamma(0,y+i,z+i) = (1,y'+i,z'-ir)$. In which case, the 4-cycle

$$(0,y+1,z)(0,y,z+1)(1,y,z)(0,y,z-1)$$

is mapped to

$$(1,y'+1,z')(1,y',z'\pm r)\gamma(1,y,z)(1,y',z'\pm r).$$

Therefore, we must have $\gamma(1,y,z) = (0,y',z')$.

Lemma 7 *Let* ρ, δ *and* α *be as in* (10), (11) *and* (12)*. Then* $(\mathrm{Aut}\Gamma(n,r))_M = \langle\rho,\alpha\rangle$.

Proof Let $\gamma \in (\mathrm{Aut}\Gamma(n,r))_M$ interchange L and R. By Lemma 6, we can compose γ with some power of ρ such that $(1,0,0)$ and $(0,0,0)$ are interchanged. We may also compose with δ such that (0,1,1) is mapped to (1,1,r). It then follows that $(0,i,i)$ is mapped to $(1,i,ir)$ and $(1,i,i)$ is mapped to $(0,i,ir)$. Thus, we have that $\gamma\rho^k\delta^h = \alpha$ for some integers k and h. Since $\alpha^2 = \delta$, we have that $(\mathrm{Aut}\Gamma(n,r))_M = \langle\rho,\alpha\rangle$.

There may exist automorphisms which do not fix M. In order to fully determine $\mathrm{Aut}\Gamma(n,r)$, we need to identify these automorphisms if they exist. We will need the following lemma.

Lemma 8 $\Gamma(n,r)$ *is edge-transitive if and only if* $(\mathrm{Aut}\Gamma(n,r))_M$ *is a proper subgroup of* $\mathrm{Aut}\Gamma(n,r)$.

Proof It is enough to show that $\Gamma(n, r)$ is edge-transitive if $(\mathrm{Aut}\Gamma(n, r))_M$ is a proper subgroup of $(\mathrm{Aut}\Gamma(n, r))$.

$(\mathrm{Aut}\Gamma(n, r))_M$ has two edge orbits. If $\mathrm{Aut}\Gamma(n, r)$ is not edge-transitive then it has at least two edge orbits. Therefore, $\mathrm{Aut}\Gamma(n, r)$ coincides with $(\mathrm{Aut}\Gamma(n, r))_M$, contradicting the fact that $(\mathrm{Aut}\Gamma(n, r))_M$ is a proper subgroup of $\mathrm{Aut}\Gamma(n, r)$.

We now determine when $\Gamma(n, r)$ is edge-transitive. Let C be a cycle in $\Gamma(n, r)$. As in [3], we define the following parameters:

$$\begin{aligned} l(C) &:= \text{ number of edges in } C \cap L; \\ r(C) &:= \text{ number of edges in } C \cap R; \\ m(C) &:= \text{ number of edges in } C \cap M. \end{aligned}$$

Let Δ be the set of all 4-cycles in $\Gamma(n, r)$, and further define

$$L_4 := \sum_{C \in \Delta} l(C); \tag{13}$$

$$R_4 := \sum_{C \in \Delta} r(C); \tag{14}$$

$$M_4 := \sum_{C \in \Delta} m(C). \tag{15}$$

Lemma 9 *Let L_4, R_4 and M_4 be as in (13), (14) and (15). Then $4L_4 = 4R_4 = M_4$ if $(\mathrm{Aut}\Gamma(n, r))_M \neq \mathrm{Aut}\Gamma(n, r)$.*

Proof By Lemma 8, $\Gamma(n, r)$ is edge-transitive. Let e be an edge involved in k 4-cycles. By edge-transitivity, every edge is involved in k 4-cycles. Since we have $2n$ edges in L, $2n$ edges in R and $8n$ edges in M, we have that $L_4 = 2nk$, $R_4 = 2nk$ and $M_4 = 8nk$. Therefore, $4L_4 = 4R_4 = M_4$.

In view of Lemma 9, we will now count the number of 4-cycles in $\Gamma(n, r)$. We distinguish between different types of 4-cycles for ease of counting. We therefore partition Δ, the set of all 4-cycles in $\Gamma(n, r)$, as follows:

$$A_1 := \{C \in \Delta : |C \cap M_1| = 4\}; \tag{16}$$

$$A_2 := \{C \in \Delta : |C \cap M_2| = 4\}; \tag{17}$$

$$B_1 := \{C \in \Delta : |C \cap M_1| = 2 \text{ and } |C \cap R| = 2\}; \tag{18}$$

$$B_2 := \{C \in \Delta : |C \cap M_2| = 2 \text{ and } |C \cap R| = 2\}; \tag{19}$$

$$D_1 := \{C \in \Delta : |C \cap M_1| = 2 \text{ and } |C \cap L| = 2\}; \tag{20}$$

$$D_2 := \{C \in \Delta : |C \cap M_2| = 2 \text{ and } |C \cap L| = 2\}; \tag{21}$$

$$E := \{C \in \Delta : |C \cap M| = 2 \text{ and } |C \cap L| = 1 \text{ and } |C \cap R| = 1\}. \tag{22}$$

We also define the sets A, B and D as:

$$A := A_1 \cup A_2; \tag{23}$$
$$B := B_1 \cup B_2; \tag{24}$$
$$D := D_1 \cup D_2, \tag{25}$$

so that A, B, D and E form a partition of Δ.

Lemma 10 *Let L_4, R_4 and M_4 be as in (13), (14) and (15). Then $4R_4 = 4L_4 \neq M_4$ in $\Gamma(n, r)$ for any n.*

Proof Let A_1, A_2, B_1, B_2, D_1, D_2, E, A, B and D be as in (16)–(25).
The vertex $(0, 0, z)$ in V_{00} is involved in the following four 4-cycles in A_1:

1. $(0, 0, z)(1, 0, z-r)(0, 0, z-r-1)(1, 0, z-1)$;
2. $(0, 0, z)(1, 0, z-1)(0, 0, z-1+r)(1, 0, z+r)$;
3. $(0, 0, z)(1, 0, z+1)(0, 0, z+1-r)(1, 0, z-r)$;
4. $(0, 0, z)(1, 0, z+r)(0, 0, z+r+1)(1, 0, z+1)$.

The second and fourth vertices in the above 4-cycles cover all four of the vertices adjacent to $(0, 0, z)$ in V_{10}. Only if the third co-ordinate in any of the third vertices above equals an element in $\{z-2, z-2r, z+2, z+2r\}$ does there exist another 4-cycle in A_1 involving $(0, 0, z)$. We solve the following equations to find possible instances where such exceptions exist. By considering parities, it is only these eight equations that we need to consider:

(i) $2 = -r-1 \Rightarrow -r = 3 \Rightarrow -1 = 9 \Rightarrow n = 5$;
(ii) $2 = r-1 \Rightarrow r = 3 \Rightarrow -1 = 9 \Rightarrow n = 5$;
(iii) $2 = -r+1 \Rightarrow r = -1$, which is impossible;
(iv) $2 = r+1 \Rightarrow r = 1$, which is impossible;
(v) $2r = -r-1 \Rightarrow 3r = 1 \Rightarrow -9 = 1 \Rightarrow n = 5$;
(vi) $2r = r-1 \Rightarrow r = -1$, which is impossible;
(vii) $2r = -r+1 \Rightarrow 3r = 1 \Rightarrow -9 = 1 \Rightarrow n = 5$;
(viii) $2r = r+1 \Rightarrow r = 1$, which is impossible.

Thus the only exceptional case is when $n = 5$. For now, we consider only $n > 5$.
If each vertex in V_{00} is involved in four 4-cycles in A_1 then we have $4n$ 4-cycles in A_1. However, in the $4n$ 4-cycles we would have counted each cycle twice since each $(0, 0, z)$ will appear again as the third vertex in the consideration of some other $(0, 0, z')$. Thus, we have $2n$ 4-cycles in A_1. It is clear that the order of A_1 equals the order of A_2, so that the order of A is $4n$.

$(0, 0, z)$ is involved in only one cycle in B_1:

1. $(0, 0, z)(1, 0, z+r)(1, 1, z)(1, 0, z-r)$.

so that the order of B_1 is n. It is easy to see that the order of B_1 equals the order of B_2, so that the order of B is $2n$. Similarly, the order of D is $2n$.
$(0, 0, z)$ is involved in eight 4-cycles in E:

Table 1 R_4, L_4 and M_4 for $n > 5$

Set	4-cycles	L_4	R_4	M_4
A	4n	0	0	16n
B	2n	0	4n	4n
D	2n	4n	0	4n
E	8n	8n	8n	16n
All	16n	12n	12n	40n

1. $(0, 0, z)(0, 1, z-1)(1, 1, z)(1, 0, z+r)$;
2. $(0, 0, z)(0, 1, z-1)(1, 1, z)(1, 0, z-r)$;
3. $(0, 0, z)(0, 1, z-1)(1, 1, z-1+r)(1, 0, z-1)$;
4. $(0, 0, z)(0, 1, z-1)(1, 1, z-1-r)(1, 0, z-1)$;
5. $(0, 0, z)(0, 1, z+1)(1, 1, z+1+r)(1, 0, z+1)$;
6. $(0, 0, z)(0, 1, z+1)(1, 1, z+1-r)(1, 0, z+1)$;
7. $(0, 0, z)(0, 1, z+1)(1, 1, z)(1, 0, z+r)$;
8. $(0, 0, z)(0, 1, z+1)(1, 1, z)(1, 0, z-r)$.

Similar to cycles in A, we determine possible exceptional cases. In this case, an exception will occur if an element of $\{z-2, z-2r, z+2, z+2r\}$ equals an element of $\{z-r-1, z-1+r, z+1-r, z+r+1, z\}$. This is the same as before except that we now need to consider when an element of $\{z-2, z-2r, z+2, z+2r\}$ equals z. Clearly, this is impossible. Thus we have that the order of E is $8n$.

Table 1 summarises R_4, L_4 and M_4 for A, B, D and E in $\Gamma(n,r)$ when $n > 5$.

Let us now consider the case $n = 5$. If $n = 5$ then we can have $r = 2$ or $r = 3$. However, it turns out that $\Gamma(5, r)$ is the same graph whether $r = 2$ or $r = 3$. This is because we have $(1, 0, r), (1, 0, -r) \in U$, and $2 \equiv -3 \pmod 5$. Thus we need only consider when $r = 2$.

We observe that each $|V_{xy}| = 5$. We count the 4-cycles in A. Each vertex (x, y, i) is adjacent to four out of five vertices in $V_{(x+1)y}$. Without loss of generality, we consider the vertex $(0, 0, 0)$ and count the 4-cycles in A which $(0, 0, 0)$ is involved in. Starting at $(0, 0, 0)$ we have four choices for the second vertex $(1, 0, z_1)$. We then have three choices for the third vertex $(0, 0, z_2)$ since we cannot trace back to $(0, 0, 0)$ or $(0, 0, z_2)$. For the fourth vertex we cannot trace back to $(1, 0, 0)$, $(1, 0, z_1)$ or $(1, 0, z_2)$ so that we have two choices for the fourth vertex $(1, 0, z_3)$. Therefore, taking double counting into account, $(0, 0, 0)$ is involved in $\frac{4\times3\times2}{2} = 12$ type A 4-cycles. The order of A is then $12 \times 10 = 120$.

We now count 4-cycles in B and D. Again we consider $(0, 0, 0)$. There are exactly three vertices in V_{11} which have in common with $(0, 0, 0)$ two incident vertices in V_{01}. Similarly, the same applies for V_{10}. Therefore, $(0, 0, 0)$ is involved in three 4-cycles in B and three in D. By similar arguments, the order of B and D equals $\frac{20\times3}{2} = 30$.

Considering now type E 4-cycles. There exist two paths from $(0, 0, 0)$ to $(1, 1, 0)$ via V_{10} and two via V_{01}, so that we have four 4-cycles in E involving both $(0, 0, 0)$ and $(1, 1, 0)$. Similarly, we have two 4-cycles in E involving $(0, 0, 0)$ and $(1, 1, 1)$, two involving $(0, 0, 0)$ and $(1, 1, 2)$, two involving $(0, 0, 0)$ and $(1, 1, 3)$, and two

Table 2 R_4, L_4 and M_4 for $n = 5$

Set	4-cycles	L_4	R_4	M_4
A	120	0	0	480
B	30	0	60	60
D	30	60	0	60
E	60	60	60	120
All	240	120	120	720

involving $(0, 0, 0)$ and $(1, 1, 4)$. In total then, we have 12 involving $(0, 0, 0)$, so that we have $12 \times 5 = 60$ type E 4-cycles in total. We summarise in Table 2.

Therefore, $4R_4 = 4L_4 \neq M_4$ for any n.

The following theorem completes our consideration of the automorphism groups.

Theorem 1 *Let ρ and α be as in* (10) *and* (12). *Then* $\mathrm{Aut}\Gamma(n, r) = \langle \rho, \alpha \rangle$ *for any n.*

Proof By Lemmas 7, 9 and 10, the result follows.

Now that we have established the automorphism groups of $\Gamma(n, r)$, applying the following theorem in Sabidussi [17] proves to be a relatively simple matter.

Theorem 2 [17] *A graph is a Cayley graph on a group if and only if the automorphism group contains a subgroup which acts regularly on the vertex set.*

Theorem 3 *$\Gamma(n, r)$ is vertex-transitive and non-Cayley on groups.*

Proof It suffices to show non-Cayleyness on groups. By Theorem 2, we require a subgroup of $\mathrm{Aut}\Gamma(n, r)$ of order $4n$ which acts transitively on $V(\Gamma(n, r))$ for Cayleyness on groups.

By Theorem 1, any automorphism in $\mathrm{Aut}\Gamma(n, r)$ is of the form $\rho^i \alpha^j$ for integers i and j. Now, if both i and j are even, then $\rho^i \alpha^j$ fixes V_{00}, V_{01}, V_{10} and V_{11}. If i is even and j is odd, then $\rho^i \alpha^j$ fixes $V_{00} \cup V_{10}$ and $V_{01} \cup V_{11}$. If i is odd and j is even, then $\rho^i \alpha^j$ fixes $V_{00} \cup V_{01}$ and $V_{10} \cup V_{11}$. Therefore, any transitive subgroup of $\mathrm{Aut}\Gamma(n, r)$ must contain an element of the form $\rho^i \alpha^j$ where both i and j are odd.

The required subgroup must have order $4n$. Now, $|\langle \alpha^j \rangle| = 4$ for any odd j. Also, no power of ρ is equal to any power of α except that $\alpha^4 = \rho^{2n} = 1$. This means that $\langle \rho^i \alpha^j \rangle = \langle \rho^i \rangle \times \langle \alpha^j \rangle$. Therefore in our required subgroup, we need some subgroup of $\langle \rho \rangle$, say $\langle \rho^i \rangle$, such that $|\langle \rho^i \rangle| = n$. The only such possibility is $\langle \rho^2 \rangle$. However there exists no ρ^i in $\langle \rho^2 \rangle$ with odd i, as required. Therefore, there exists no subgroup of $\mathrm{Aut}\Gamma(n, r)$ which acts regularly on $V(\Gamma(n, r))$.

4 Conclusion

It is expected that there exists many variations of U such that $\mathrm{Cay}(Q(n, r), U)$ is a VTNCG. If $U = \{(1, 0, 0), (0, 1, 1), (0, 1, -1)\}$ for example, $\mathrm{Cay}(Q(n, r), U)$ is isomorphic to the generalised Petersen graphs, which are also VTNCGs if $r^2 \equiv -1$

(mod n) (and thus $(n, r) = 1$). More generally, there are indications that determining meta-Cayley graphs on a product of loops and groups is a fruitful avenue in finding vertex transitive graphs which are non-Cayley on groups.

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