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RESEARCH ARTICLE

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A fitted numerical method for parabolic turning point singularly perturbed problems with an interior layer

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Abstract

The objective of this paper is to construct and analyze a fitted operator finite difference method (FOFDM) for the family of time-dependent singularly perturbed parabolic convection–diffusion problems. The solution to the problems we consider exhibits an interior layer due to the presence of a turning point. We first establish sharp bounds on the solution and its derivatives. Then, we discretize the time variable using the classical Euler method. This results in a system of singularly perturbed interior layer two-point boundary value problems. We propose a FOFDM to solve the system above. Through a rigorous error analysis, we show that the scheme is uniformly convergent of order one with respect to both time and space variables. Moreover, we apply Richardson extrapolation to enhance the accuracy and the order of convergence of the proposed scheme. Numerical investigations are carried out to demonstrate the efficacy and robustness of the scheme.

KEYWORDS

error bounds, finite difference methods, interior layer, singularly perturbed problems, uniform convergence

1 | INTRODUCTION

In this paper, we consider the turning point parabolic singularly perturbed problems with interior layer

$$Lu := -d(x,t)u_t + \varepsilon u_{xx} + a(x,t)u_x - b(x,t)u = f(x,t), -1 \le x \le 1; \quad t \in [0,T]; \tag{1.1}$$

$$u(-1,t) = \alpha, \quad u(1,t) = \gamma, \quad u_0(x) = u(x,0), \tag{1.2}$$

where α and γ are given real numbers and the perturbation parameter ε satisfies $0 < \varepsilon \ll 1$. The coefficients functions $a(x,t)$, $b(x,t)$, $d(x,t)$, $f(x,t)$ and $u_0(x)$ are assumed to be sufficiently smooth to ensure the smoothness of the solution. Also $d(x, t) > 0\ \forall (x, t) \in [-1, 1] \times [0, T]$. The condition of the reaction factor $b(x,t) \geq b_0 > 0$, $\forall\ t \in [0, T]$ ensures the uniqueness of the solution [1].

The problem (1.1) and (1.2) is said to be a turning point problem, if there exists α_i with—$1 < \alpha_i < 1$ such that $a(\alpha_i,t) = 0$ and $a(-1, t)a(1, t) \neq 0, \forall\ t \in [0, T]$. The r zeros α_i, $i = 1, 2, \ldots, r$ of $a(x, t)$ are called turning points. These statements can be seen in Berger et al. [2] where they also showed that the bounds of the solution to the problem (1.1) and (1.2) near the given turning point α_i depend on ε and the constants $\beta_i = b(\alpha_i,t)/a_x(\alpha_i,t)$. When $\beta_i < 0$, the solution to $u(x,t)$ is "smooth" near $(x,t) = (\alpha_i,t)$, and if $\beta_i > 0$, the solution $u(x,t)$ presents a rapid change at $(x, t) = (\alpha_i, t)\ \forall\ t \in [0, T]$ termed "interior layer" which is often shown up by the change in signs of the convection coefficient $a(x,t)$ near (α_i, t) $\forall (x, t) \in [-1, 1] \times [0, T]$. In the case where the convection coefficient $a(x,t)$ does not change the sign throughout the spatial domain, the boundary layer may occur near -1 or/and 1. In addition, the existence of $\alpha_0 \in [-1,1]$, such that $|a_x(x, t)| \geq |a_x(\alpha_0, t)|/2, \forall\, t \in [0, T]$, ensures the uniqueness of the turning point in $[-1,1]$.

In this paper, we consider the assumptions below to guarantee the interior layer of the solution to problem (1.1) and (1.2) at $x = 0, \forall\ t \in [0, T]$,

$$\begin{cases} a(0,t) = 0, & a_x(0,t) > 0, t \in [0,T], \\ |a_x(x,t)| \geq \frac{|a_x(0,t)|}{2}, & x \in [-1,1], t \in [0,T], \\ \frac{b(0,t)}{a_x(0,t)} > 0, & x \in t \in [0,T], \\ b(x,t) \geq b(0,t) > 0, & x \in [-1,1], t \in [0,T]. \end{cases} \tag{1.3}$$

The interior layers may also originate from discontinuous data [3–5].

Parameter-sensitive problems such as (1.1) and (1.2) in which the perturbation parameter multiplies the highest derivative of the underlying differential equation are termed singularly perturbed problems. They have attracted researchers' attention over the last few decades because of the existence of oscillations or spurious solutions when trying to solve them numerically. These challenges are more pronounced as the parameter approaches zero and classical numerical methods fail to generate suitable approximations to the solution.

In the context of finite difference discretizations, two families of methods are widely used namely the fitted mesh finite difference methods (see e.g., [6–8]) and the fitted operator finite difference methods (FOFDM) [9–11].

Recently, a very large number of special methods have been developed by various authors to solve nonturning and turning points time dependent singularly perturbed parabolic problems using implicit Euler method for time discretization. Some authors developed appropriate spatial discretizations adapted to the conditions of their problems. For instance [12] developed finite difference schemes using a semi-discrete Petrov–Galerkin finite element method. In Clavero et al. [13] an upwind finite difference scheme is derived, and [14] constructed an upwind and midpoint upwind difference methods for the discretization of space variable. In Kadalbajoo et al. [15] a B-spline collocation method is designed. Readers who need more information related to nonturning points time dependent singularly perturbed parabolic problems may refer to [16–19], and those who are interested in time dependent singularly perturbed parabolic problems when the turning points lead to boundary and/or interior layer(s) are referred to [20–24].

Discussions on fitted finite difference methods to solve time dependent singularly perturbed convection–diffusion problems whose solution exhibits an interior layer are rare. Nevertheless, we have for instance Clavero et al. [25] who developed a classical upwind finite difference scheme on a piecewise defined mesh of Shishkin type to solve a one-dimensional parabolic singularly perturbed reaction–diffusion problems with parameters affecting the diffusion and the convection terms. Dunne and O'Riordan [26] constructed numerical methods involving piecewise uniform meshes of Shishkin type which fitted to interior and boundary layers. The methods were used to solve singularly perturbed parabolic problems in which the coefficients are discontinuous in the space variable. O'Riordan and Quinn [27] examined a linear time dependent singularly perturbed convection–diffusion problems, where the convective coefficient got interior layer; to design and analyze a monotone finite difference operator and a piecewise-uniform Shishkin mesh. Gracia and O'Riordan [28] constructed and analyzed a numerical method consisting on a monotone finite difference operator and piecewise uniform mesh. This method was used to solve a linear singularly perturbed time dependent convection–diffusion problem, in which initial condition was designed to have steep gradient in the vicinity of the inflow, transported in time to create a moving interior chock layer.

In several works on time dependent problems, as we can notice from the references above, in the discretization of interior layer problems based on difference equation theory [29], there has never been singularly perturbed problem with smooth coefficients depending on both space and time variables.

The main aim of this paper is to construct and analyze a FOFDM based difference equation theory and implicit Euler method to obtain piecewise uniform meshes respectively on space and time. This strategy approximates the solution of time dependent singularly perturbed problems (1.1) and (1.2), where the coefficients are functions of space and time variables and the solution to the problem exhibits an interior layer due to the presence of a turning point. We show that the method converges uniformly of order one in both space and time variables. We also use Richardson extrapolation [6, 23], as the acceleration technique to improve the accuracy and the order of convergence of the FOFDM designed up to order two in space only.

The paper has been organized as follows: in Section 2 we provide qualitative results on the bounds of the solution and its derivatives at every time level t in $[0,T]$. Using techniques (tools) presented in [2, 13, 30], we then provide sharp error estimates specific to the class of problems (1.1) and (1.2). And Section 3 presents some a priori estimates on time discretization. In Section 4, we introduce the proposed scheme which is analyzed in Section 5. Section 6 deals with Richardson extrapolation. To show the effectiveness of the proposed scheme, we carry out and discuss some numerical experiments in Section 7. Section 8 is devoted to some concluding remarks.

2 | QUALITATIVE RESULTS

In this section, some results related to the continuous problem are presented. We use these results in Section 5 of the error analysis. $f(x,t)$ and $u(x,0)$ are herein assumed to be smooth functions to secure the continuity and ε-uniform bound of the solution with its derivatives to the problem (1.1) and (1.2). These conditions are required for appropriate space and time accuracy when using the maximum norm on the domain $\overline{D} = \overline{\Omega} \times [0, T]$, with $\Omega = (-1,1)$ and $D = \Omega \times (0,T]$.

Lemma 2.1 (Minimum principle). *Let ψ be a smooth function satisfying $\psi(-1,t) \geq 0$, $\psi(1,t) \geq 0$, $\forall t \in [0, T]$ and $L\psi(x, t) \leq 0$, $\forall\, (x, t) \in D$. Then, $\psi(x, t) \geq 0$, $\forall (x, t) \in \overline{D}$.*

Proof. Assume that there exists a point $(x^*, t^*) \in \overline{D}$ such that $\psi(x^*, t^*) = \min\psi(x, t) < 0$. It follows that (x^*,t^*) cannot be of the form $(-1, t)$, $(1, t)$ or $(x, 0)$. From the definition, $\psi_x(x^*, t^*) = 0$, $\psi_t(x^*, t^*) = 0$ and $\psi_{xx}(x^*, t^*) \geq 0$. We also have

$$L\psi(x^*, t^*) = \varepsilon\psi_{xx}(x^*, t^*) + a(x^*, t^*)\psi_x(x^*, t^*) - b(x^*, t^*)\psi(x^*, t^*) + \psi_t(x^*, t^*) > 0,$$

which is false. It follows that $\psi(x^*,t^*) \geq 0$, and thus $\psi(x, t) \geq 0$, $\forall (x, t) \in \overline{D}$. ∎

We use this minimum principle to prove Lemma 2.2.

Lemma 2.2 (Uniform stability estimate). *Let $u(x,t)$ be the solution of* (1.1) and (1.2). *Then, we have*

$$\|u(x,t)\| \leq C(b_0^{-1}\|f(x,t)\| + \max(|\alpha|,|\gamma|)), \quad \forall (x,t) \in \overline{D},$$

where $\|.\|$ denotes the maximum norm on the domain $\overline{D}$, and b_0 a positive constant as specified above in the introduction.

Proof. Consider the comparison function

$$\Pi^{\pm}(x,t) = b_0^{-1}\|f(x,t)\| + \max(|\alpha|,|\gamma|) \pm u(x,t), \quad x \in \overline{D}.$$

We have

$$L\Pi^{\pm}(x,t) = -\frac{b(x,t)}{b_0}\|f(x,t)\| - b(x,t)\max(|\alpha|,|\gamma|) \pm Lu(x,t) \leq 0.$$

Using the minimum principle above it follows that

$$\Pi^{\pm}(x,t) \geq 0, \quad \forall (x,t) \in \overline{D}.$$

Consequently

$$\|u(x,t)\| \leq C(b_0^{-1}\|f(x,t)\| + \max((|\alpha|,|\gamma|))), \quad \forall (x,t) \in \overline{D},$$

which completes the proof. ∎

For the rest of this work we consider the following partition of $\overline{\Omega} = [-1,1]$: $\Omega_L = [-1,-\delta)$, $\Omega_C = [-\delta,\delta]$, $\Omega_R = (\delta,1]$, where $0 < \delta \leq 1/2$. Furthermore, $\Omega_C = \Omega_C^- \cup \Omega_C^+$, with $\Omega_C^- = [-\delta,0)$, $\Omega_C^+ = [0,\delta]$ and $\overline{D} = \overline{\Omega} \times [0,T]$.

Lemmas 2.3 and 2.4 provide the appropriate bounds on the solution to the problem (1.1) and (1.2) and its derivatives, depending on whether x belongs to Ω_L, Ω_C, or Ω_R.

It is well known that if $u(x,t)$ is the solution to the problem (1.1) and (1.2), then there exists a positive constant C such that $|u(x,t)| \leq C$, $\forall (x,t) \in \overline{D}$.

Lemma 2.3 *Let $u(x,t)$ be the solution to* (1.1) and (1.2) *and $a(x,t)$, $b(x,t)$ and $f(x,t)$ sufficiently smooth functions in $\overline{D}$. Then, there exists a constant C such that*

$$\left|\frac{\partial^i u(x,t)}{\partial x^i}\right| \leq C, \quad \forall (x,t) \in \overline{D} \setminus \Omega_C.$$

Proof. See [13]. ∎

Lemma 2.4 *Let $u(x,t)$ be the solution to* (1.1) and (1.2) *and $a(x,t)$, $b(x,t)$, and $f(x,t)$ sufficiently smooth functions in $\overline{D}$. Then there exist positive constants η and C such that*

$$\left|\frac{\partial^i u(x,t)}{\partial x^i}\right| \leq C\left[1 + \varepsilon^{-i}\exp\left(\frac{\eta x}{\varepsilon}\right)\right], \quad \forall x \in \Omega_C^-, \ \ t \in [0,T], \ \ i = 0,1,2,$$

and

$$\left|\frac{\partial^i u(x,t)}{\partial x^i}\right| \leq C\left[1 + \varepsilon^{-i}\exp\left(\frac{-\eta x}{\varepsilon}\right)\right], \quad \forall x \in \Omega_C^+, \ \ t \in [0,T], \ \ i = 0,1,2.$$

Proof. We prove this lemma on Ω_C^-. The proof on Ω_C^+ can be done in similar manner. To start let us rewrite Equation (1.1) as follows:

$$L_{x,\varepsilon}u = d(x,t)\frac{\partial u}{\partial t} + f(x,t) = g(x,t), \quad \forall x \in \Omega_C^-, \quad t \in [0,T], \tag{2.1}$$

where

$$L_{x,\varepsilon}u = \varepsilon\frac{\partial^2 u}{\partial x^2} + a(x,t)\frac{\partial u}{\partial x} - b(x,t)u,$$

Assuming $u_0 = u(x,0)$, d and f smooth functions, then $g(x,t)$ is continuous and ε-uniformly bounded. We use the technique of [19] and Equation (2.1), to get

$$\left|\frac{\partial^i u(x,t)}{\partial x^i}\right| \leq C\left[1 + \varepsilon^{-i}\exp\left(\frac{\eta x}{\varepsilon}\right)\right], \quad \forall x \in \Omega_C^-, \quad t \in [0,T], \quad i = 0, 1. \tag{2.2}$$

To deduce the similar bounds for higher values of i, we consider $v(x,t) = \partial u(x,t)/\partial x$, and after differentiating (2.1) with respect to x, it follows that $\forall x \in \Omega_C^-, \ t \in [0,T]$;

$$-d(x,t)\frac{\partial v(x,t)}{\partial t} + L_{x,\varepsilon}v = m(x,t) = \frac{\partial f(x,t)}{\partial x} + \frac{\partial d(x,t)}{\partial x}\frac{\partial u}{\partial t} - \frac{\partial a(x,t)}{\partial x}\frac{\partial u}{\partial t} + \frac{\partial b(x,t)}{\partial x}u,$$

$$v(-1,t) = \frac{\partial u(-1,t)}{\partial x} = \alpha_1, \quad v(1,t) = \frac{\partial u(1,t)}{\partial x} = \gamma_1, \quad v_0(x) = \frac{\partial u(x,0)}{\partial x}$$

Assuming $m(x,t)$ smooth function and applying the above technique for the second time, yields

$$\left|\frac{\partial v}{\partial x}\right| \leq C\left[1 + \varepsilon^{-1}\exp\left(\frac{\eta x}{\varepsilon}\right)\right], \quad \forall x \in \Omega_C^-, \quad t \in [0,T],$$

which is a bound for $\partial^2 u/\partial x^2$. ■

3 | TIME DICRETIZATION

In this section, we discretize the problem (1.1) and (1.2) with respect to time, with uniform step size τ, using Euler implicit method. The partition of the time interval [0, T] is given by:

$$\overline{\omega}^k = \{t_k = k\tau, \ 0 \leq k \leq K, \ \tau = T/K\}. \tag{3.1}$$

And the discretization of the problem (1.1) and (1.2) on $\overline{\omega}^k$

$$-d(x,t_k)\frac{u(x,t_k) - u(x,t_{k-1})}{\tau} + L_{x,\varepsilon}(u(x,t_k)) = f(x,t_k), \quad 1 \leq k \leq K, \tag{3.2}$$

$$u(x,t_0) = u_0(x), \quad \forall x \in (-1,1), \quad u(-1,t_k) = \alpha, \quad u(1,t_k) = \gamma. \tag{3.3}$$

Equation (3.2) can also be written as:

$$(-d(x,t_k)I + \tau L_{x,\varepsilon})(u(x,t_k)) = \tau f(x,t_k) - d(x,t_k)u(x,t_{k-1}). \tag{3.4}$$

The discretization above is the result of the turning point singularly perturbed problems, at each time level $t_k = k\tau$ which will be examined later. The global error E_k at the time level t_k is the sum of local errors e_k at each time level t_k. This local truncation error e_k is given as: $e_k = u(x,t_k) - \tilde{u}(x,t_k)$, where $\tilde{u}(x,t_k)$ is the solution of

$$(-d(x,t)I + \tau L_{x,\varepsilon})(u(x,t_k)) = \tau f(x,t_k) - d(x,t)u(x,t_{k-1}), u(-1,t_k) = \alpha, \ u(1,t_k) = \gamma. \tag{3.5}$$

We find out that the operator $(-d(x,t)I + \tau L_{x,\varepsilon})$ verifies the maximum principle leading to:

$$\|(-d(x,t_k)I + \tau L_{x,\varepsilon})^{-1}\| \leq \frac{1}{\max_{0\leq k\leq K,\, x\in[-1,1]}(|d(x,t_k)|^{order(I)}) + \tau\beta}. \tag{3.6}$$

where *order*(I) in the inequality above is the order of the identity matrix I. Which proves the stability of the discretization with respect to time.

It is also known that the local error and the global error are respectively bounded as follows: $\|e_k\|_\infty \le c\tau^2$, $1 \le k \le K$ and $\|E_k\|_\infty \le c\tau, 1 \le k \le K$.

Lemma 3.1 *Let $u(x,t_k)$ be the solution of* (3.2) and (3.3) *at time level t_k, Then there exists a positive constant C such that*

$$|u^{(m)}(x,t_k)| \le C\left[1+\varepsilon^{-m}\exp\left(\frac{\eta x}{\varepsilon}\right)\right], \quad m=0,1,2,3, \quad \forall\ x \in \Omega_C^-,$$

and

$$|u^{(m)}(x,t_k)| \le C\left[1+\varepsilon^{-m}\exp\left(\frac{-\eta x}{\varepsilon}\right)\right], \quad m=0,1,2,3, \quad \forall\ x \in \Omega_C^+.$$

Proof. See [13]. ∎

In the next section we introduce the scheme which we analyze in a subsequent section.

4 | THE SCHEME

Let n be a positive and even integer and let us denote by $\overline{\Omega}^n$ the following partition of the interval $[-1,1]$:

$$x_0=-1;\ \ x_j=x_0+jh;\ \ j=1,\dots,n-1,\ \ h=x_j-x_{j-1},\ \ x_n=1.$$

Let $\overline{Q}^{n,K} = \overline{\Omega}^n \times \overline{\omega}^K$ be the grid of (x_j,t_k). To simplify, we adopt the following; $\forall\ (x_j,t_k) \in \overline{Q}^{n,K}$, $\Xi(x_j,t_k) := \Xi_j^k$. And U_j^k the approximation of u_j^k. Using difference equation theory on $\overline{Q}^{n,K}$ [29], we discretize the problem (1.1) and (1.2) as:

$$L^{n,K}U_j^k := \begin{cases} \varepsilon\delta^2 U_j^k + \widetilde{a}_j^k D^- U_j^k - \left(\widetilde{b}_j^k + \frac{\widetilde{d}_j^k}{\tau}\right)U_j^k = \widetilde{f}_j^k - \widetilde{d}_j^k \frac{U_j^{k-1}}{\tau}, \\ j=1,2,\dots,\frac{n}{2}-1,\ \ k=1,\dots,K, \\ \\ \varepsilon\delta^2 U_j^k + \widetilde{a}_j^k D^+ U_j^k - \left(\widetilde{b}_j^k + \frac{\widetilde{d}_j^k}{\tau}\right)U_j^k = \widetilde{f}_j^k - \widetilde{d}_j^k \frac{U_j^{k-1}}{\tau}, \\ j=\frac{n}{2},\frac{n}{2}+1,\frac{n}{2}+2,\dots,n-1,\ \ k=1,\dots,K, \end{cases} \tag{4.1}$$

$$U_0^k=\alpha,\ \ U_n^k=\gamma, \tag{4.2}$$

where

$$D^-U_j^k = \frac{U_j^k - U_{j-1}^k}{h}, \quad D^+U_j^k = \frac{U_{j+1}^k - U_j^k}{h}, \quad \delta^2 U_j^k = \frac{U_{j+1}^k - 2U_j^k + U_{j-1}^k}{\widetilde{\phi_j^k}^2},$$

and

$$\widetilde{\phi_j^k}^2 = \begin{cases} \frac{h\varepsilon}{\widetilde{a}_j^k}\left[\exp\left(\frac{\widetilde{a}_j^k h}{\varepsilon}\right)-1\right],\ \ j=1,2,\dots,\frac{n}{2}-1, \\ \\ \frac{h\varepsilon}{\widetilde{a}_j^k}\left[1-\exp\left(\frac{-\widetilde{a}_j^k h}{\varepsilon}\right)\right],\ \ j=\frac{n}{2},\frac{n}{2}+1,\frac{n}{2}+2,\dots,n-1. \end{cases} \tag{4.3}$$

Also, we have adopted the following convention for $k = 1, \ldots, K$.

$$\begin{cases} \widetilde{a}_j^k = \frac{a_j^k + a_{j-1}^k}{2} \ \text{ for } \ j = 0, 1, 2, \ldots, \frac{n}{2} - 1, \\ \widetilde{a}_j^k = \frac{a_j^k + a_{j+1}^k}{2} \ \text{ for } \ j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n - 1, \\ \widetilde{b}_j^k = \frac{b_{j-1}^k + b_j^k + b_{j+1}^k}{3}, \widetilde{f}_j^k = \frac{f_{j-1}^k + f_j^k + f_{j+1}^k}{3} \ \text{ for } \ j = 1, 2, \ldots, n-1, \\ \widetilde{F}_j^k = \widetilde{f}_j^k - \widetilde{d}_j^k \frac{U_j^{k-1}}{\tau}, \text{for } \ j = 1, 2, \ldots, n-1, \\ \widetilde{d}_j^k = \frac{d_{j-1}^k + d_j^k + d_{j+1}^k}{3}, \ \text{ for } \ j = 0, 1, 2, \ldots, n-1. \end{cases} \tag{4.4}$$

We rewrite (4.1) as

$$\begin{cases} r_{j,k}^{-} U_{j-1}^k + r_{j,k}^c U_j^k + r_{j,k}^{+} U_{j+1}^k = \widetilde{f}_j^k, \ \ j = 0, 1, 2, \ldots, \frac{n}{2} - 1; \\ k = 0, 1, \ldots, K, \\ r_{j,k}^{-} U_{j-1}^k + r_{j,k}^c U_j^k + r_{j,k}^{+} U_{j+1}^k = \widetilde{f}_j^k, \ \ j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n-1; \\ k = 0, 1, \ldots, K. \end{cases} \tag{4.5}$$

where

$$\begin{cases} r_{j,k}^{-} = \frac{\varepsilon}{\widetilde{\phi_j^k}^2} - \frac{\widetilde{a}_j^k}{h}; \ \ r_{j,k}^c = \frac{-2\varepsilon}{\widetilde{\phi_j^k}^2} + \frac{\widetilde{a}_j^k}{h} - \left(\widetilde{b}_j^k + \frac{\widetilde{d}_j^k}{\tau} \right); \ \ r_{j,k}^{+} = \frac{\varepsilon}{\widetilde{\phi_j^k}^2}, \ \ j = 0, 1, 2, \ldots, \frac{n}{2} - 1, \\ r_{j,k}^{-} = \frac{\varepsilon}{\widetilde{\phi_j^k}^2}; \ \ r_{j,k}^c = \frac{-2\varepsilon}{\widetilde{\phi_j^k}^2} - \frac{\widetilde{a}_j^k}{h} - \left(\widetilde{b}_j^k + \frac{\widetilde{d}_j^k}{\tau} \right); \ \ r_{j,k}^{+} = \frac{\varepsilon}{\widetilde{\phi_j^k}^2} + \frac{\widetilde{a}_j^k}{h}, \ \ j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n-1. \end{cases} \tag{4.6}$$

The FOFDM (4.5) along with the boundary conditions (4.2) satisfies Lemmas 4.1 and 4.2:

Lemma 4.1 (Discrete minimum principle). *For any mesh function ξ_j^k such that, $L^{n,k} \xi_j^k \leq 0 \ \forall (j,k) \in Q^{n,K}$, $\xi_j^0 \geq 0$, $0 \leq j \leq n$, $\xi_0^k \geq 0$, and $\xi_n^k \geq 0$, $1 \leq k \leq K$. Then $\xi_j^k \geq 0$, $\forall (j,k) \in \overline{Q}^{n,K}$.*

Proof. Let (s,l) be such that $\xi_s^l = \min_{(j,k)} \xi_j^k < 0$, $\xi_j^k \in \overline{Q}^{n,K}$. It is clear that $s \neq 1, 2, \ldots, n-1$ and $l \neq 1, 2, \ldots, K$; otherwise $\xi_s^l \geq 0$. Also $\xi_{s+1}^l - \xi_s^l \geq 0$, $\xi_s^l - \xi_{s-1}^l \leq 0$, and $\xi_s^l - \xi_s^{l-1} \leq 0$. We have

$$L^{n,K} \xi_s^l = \begin{cases} \varepsilon \overline{\delta}^2 \xi_s^l + a_s^l D^{-} \xi_s^l - \left(b_s^l + \frac{d_s^l}{\tau} \right) \xi_s^l > 0, \quad s = 1, 2, \ldots, \frac{n}{2} - 1, \ \ l = 1, 2, \ldots, K, \\ - \left(b_s^l + \frac{d_s^l}{\tau} \right) \xi_s^l > 0, \quad s = \frac{n}{2}, \ \ l = 1, 2, \ldots, K, \\ \varepsilon \overline{\delta}^2 \xi_s^l + a_s^l D^{+} \xi_s^l - \left(b_s^l + \frac{d_s^l}{\tau} \right) \xi_s^l > 0, \quad s = \frac{n}{2} + 1, \ldots, n-1 \ , \ \ l = 1, 2, \ldots, K. \end{cases} \tag{4.7}$$

Thus $L^{n,K} \xi_k^l > 0, s = 1, 2, \ldots, n-1$ and $l = 1, 2, \ldots, K$, which is a contradiction. It follows that $\xi_s^l \geq 0$, and thus $\xi_j^k \geq 0, \ \forall \ (j,k) \ \in \ \overline{Q}^{n,K}$.

The above minimum principle is used to prove Lemma 4.2. ∎

Lemma 4.2 (Uniform stability estimate). *Let Z_j^k be a mesh function at a time level such that $Z_0^k = Z_n^k = 0$. Then*

$$|Z_j^k| \le \frac{1}{b_0} \max_{1\le i\le n-1} |L^{n,K} Z_i^k|, \ \ for \quad 1 \le j \le n, and \ \ 1 \le k \le K.$$

Proof. Consider the mesh function

$$(\xi^{\pm})_j^k = \frac{1}{b_0} \max_{1\le i\le n-1} |L_\varepsilon^{n,K} Z_i^k| \pm Z_j^k, \quad 1 \le j \le n, \ \text{ and } \ 1 \le k \le K,$$

with $b_j^k \ge b_0 > 0$ to ensure the uniqueness of the solution to the problem (4.1) and (4.2). It is clear that $(\xi^{\pm})_0^k \ge 0$ and $(\xi^{\pm})_n^k \ge 0$. Also, for $0 \le j \le n$, and $1 \le k \le K$,

$$L^{n,K}(\xi^{\pm})_j^k = \frac{-b_j^k}{b_0} \max_{1\le i\le n-1} |L^{n,K} Z_i^k| \pm L^{n,K} z_j^k, \quad 1 \le j \le n, \ \text{ and } \ 1 \le k \le K.$$

For $0 \le j \le n, (-b_j^k)/(b_0) \le -1$. This leads to $L^{n,K}(\xi^{\pm})_j^k \le 0$. By the discrete minimum principle (Lemma 4.1), we conclude that $(\xi^{\pm})_j^k \ge 0, \ \forall \ 0 \le j \le n, 1 \le k \le K$ and this ends the proof. ∎

Lemma 4.3 *For a fixed mesh and for all integers m, we have*

$$\lim_{\varepsilon->0} \max_{1\le j\le \frac{n}{2}-1} \frac{\exp(Mx_j/\sqrt{\varepsilon})}{\varepsilon^{m/2}} = 0, and \lim_{\varepsilon->0} \max_{\frac{n}{2}\le j\le n-1} \frac{\exp(-Mx_j/\sqrt{\varepsilon})}{\varepsilon^{m/2}} = 0.$$

Proof. See [8]. ∎

In the next section we concentrate on convergence analysis of the FOFDM derived.

5 | CONVERGENCE ANALYSIS OF FOFDM

In this section we analyze the FOFDM described in the previous section. The analysis will be conducted on $x \in [-1,0]$ and the case when $x \in (0,1]$ can be done similarly.

Let us define the operator L^K from (3.3) as:

$$\begin{aligned} L^K z(x,t_k) &= \varepsilon \frac{d^2 z(x,t_k)}{dx^2} + a(x,t_k)\frac{dz(x,t_k)}{dx} - \left(b(x,t_k) + \frac{d(x,t_k)}{\tau}\right) z(x,t_k), \\ &= f(x,t_k) - d(x,t_k)\frac{z(x,t_{k-1})}{\tau}. \end{aligned} \tag{5.8}$$

The local truncation error of the space discretization on $[-1,0]\times[0,T]$ (e.g., $j = 1, 2, \dots, n/2–1$, $k = 1, 2, \dots, K$) can be given as:

$$\begin{aligned} L^{n,K}(U_j^k - z_j^k) &= (L^K - L^{n,K}) z_j^k \\ &= \varepsilon z''_{j,k} + \widetilde{a}_j^k z_j^k - \left[\frac{\varepsilon}{\widetilde{\phi_j^2}^k}(z_{j+1}^k - 2z_j^k + z_{j-1}^k) + \frac{\widetilde{a}_j^k}{h}(z_j^k - z_{j-1}^k) \right] \\ &= \varepsilon u''_{j,k} - \frac{\varepsilon}{\widetilde{\phi_j^2}^k}\left[h^2 u''_{j,k} + \frac{h^4}{24}(z^{(iv)})^k(\xi_1) + \frac{h^4}{24}(z^{(iv)})^k(\xi_2) \right] \\ &\quad + \frac{\widetilde{a}_j^k h}{2} z''_{j,k} - \frac{\widetilde{a_j}^k h^2}{6} z'''_{j,k} + \frac{\widetilde{a_j}^k h^3}{24}(z^{(iv)})^k(\xi_3), \end{aligned} \tag{5.9}$$

with $\xi_1 \in (x_j,x_{j+1})$, $\xi_2,\xi_3 \in (x_{j-1},x_j)$. Using the expression for $\widetilde{a}_j^k$ in reference to (4.4), the Taylor expansions of a_{j-1}^k up to order four, and the truncated Taylor expansion $1/\widetilde{\phi_j^2}^k = 1/h^2 - \widetilde{a}_j^k/\varepsilon h$, it follows that

$$L_1^{n,K}(U_j^k - z_j^k) = \frac{3}{2}a_j^k u''_{j,k}h! + \left[-\frac{3a'_{j,k}}{2}z''_{j,k} - \frac{\varepsilon}{24}((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)) - \frac{a_j^k}{6}z'''_{j,k}\right]h^2$$
$$+\left[\frac{3a''_{j,k}}{4}z''_{j,k} - \frac{a_j^k}{24}((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)) + \frac{a'_{j,k}}{12}z'''_{j,k} + \frac{a_j^k}{24}(z^{(iv)})^k(\xi_3)\right]h^3$$
$$+\left[-\frac{13a'''_{j,k}}{24}z''_{j,k} - \frac{a'_{j,k}}{48}((u^{(iv)})^k(\xi_1)) + (z^{(iv)})^k(\xi_2)) - \frac{a''_{j,k}}{24}z'''_{j,k} - \frac{a'_{j,k}}{48}(z^{(iv)})^k(\xi_3)\right]h^4. \tag{5.10}$$

where ξ's lie in the interval (x_{j-1},x_{j+1}). Note that the coefficients of $u_j^k, z'_{j,k}, \dots, (z^{(iv)})^k(\xi_{*_j})$ can be bounded by a constant. Now, applying Lemmas 3.1 and 4.3 it follows that

$$|L_1^{n,K}(U_j^k - z_j^k)| \le Mh, \quad \forall j = 1(1)\frac{n}{2} - 1.$$

In a similar way, we can prove that

$$|L_2^{n,K}(U_j^k - z_j^k)| \le Mh, \quad \forall j = \frac{n}{2}(1)n + 1.$$

Lemma 4.2, leads to the following results.

Theorem 5.1 *Let U_j^k be the numerical solution of* (4.1) *along with* (4.4) *and z_j^k the solution to* (3.2) and (3.3) *at time level t_k. Then, there exists a constant M independent of ε, τ, h and k such that*

$$\max_{1\le j\le n+1} |U_j^k - z_j^k| \le Mh \quad k = 1(1)K + 1. \tag{5.11}$$

Triangular inequality $|U_j^k - u_j^k| \le |U_j^k - z_j^k| + |z_j^k - u_j^k|$ along with Lemma 4.2, Theorem 5.1 and the global error; lead to the following main result.

Theorem 5.2 *Let U_j^k be the numerical solution of* (4.1)–(4.4) *and u_j^k the solution to* (1.1) and (1.2) *at the grid point (x_j,t_k). Then, there exists a constant M independent of ε, τ, h and k such that*

$$\max_{0\le j\le n} |U_j^k - u_j^k| \le M(h + \tau). \tag{5.12}$$

In the next section we deal with Richardson extrapolation which is an acceleration technique. We use this technique to improve the estimate (5.12).

6 | RICHARDSON EXTRAPOLATION ON FOFDM

Richardson extrapolation is the extrapolation technique based on linear combination of p solutions, $p \ge 0$ corresponding to different, nested meshes.

In this section we improve the accuracy and the order of convergence of (5.12). To begin, we look back to (5.10) that can also be written as:

$$L^{n,K}(U_j^k - z_j^k) = M_1 h + M_2 h^2 + R_n(x_j), \tag{6.1}$$

where

$$
\begin{aligned}
M_1 &= \frac{3a_j}{2} z''_{j,k},\\
M_2 &= \frac{3a'_{j,k}}{3} - \frac{\varepsilon}{24}((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)) - \frac{a_j^k}{6} z'''_{j,k},\\
R_n^k(x_j) &= h^3 \left[\frac{3a''_{j,k}}{4} z''_{j,k} - \frac{a_j^k}{24}((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)) + \frac{a'_{j,k}}{12} z'''_{j,k} + \frac{a_j^k}{24}(z^{(iv)})^k(\xi_3) \right]\\
&\quad + h^4 \left[\frac{13a'''_{j,k}}{24} z''_{j,k} - \frac{a'_{j,k}}{48}((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)) - \frac{a''_{j,k}}{24} z'''_{j,k} - \frac{a'_{j,k}}{48}(z^{(iv)})^k(\xi_3) \right].
\end{aligned}
$$

The ξ's and $z_j^k, z'_{j,k}, \ldots, (z^{(iv)})^k(\xi_{*_j})$ remain the same as specified in (5.9). Now, let μ_{2n} be the mesh obtained by bisecting each mesh interval in μ_n, that is,

$$\mu_{2n} = \{\overline{x}_i\} \quad \text{with} \quad \overline{x}_0 = -1, \quad \overline{x}_n = 1 \quad \text{and} \quad \overline{x}_j - \overline{x}_{j-1} = \overline{h} = h/2, \quad j = 1, 2, \ldots, 2n.$$

$\overline{U}_j^k$ the numerical solution on μ_{2n}. M and p positive constants. Equation (6.1) can be written in terms of $\overline{U}_j^k$ as follows:

$$L^{n,K}(\overline{U}_j^k - \overline{z}_j^k) = M\overline{h} + p\overline{h}^2 + R_{2n}^k(\overline{x}_j), 1 \le j \le 2n-1. \tag{6.2}$$

Note that $\overline{z}_j^k \equiv z_j^k$.

Multiplying (6.2) by 2, leads to

$$2L^{n,K}(\overline{U}_j^k - \overline{z}_j^k) = 2M\overline{h} + 2p\overline{h}^2 + 2R_{2n}^k(\overline{x}_j), 1 \le j \le 2n-1, \tag{6.3}$$

or

$$L^{n,K}(2\overline{U}_j^k - 2\overline{z}_j^k) = 2M\overline{h} + 2p\overline{h}^2 + 2R_{2n}^k(\overline{x}_j), 1 \le j \le 2n-1. \tag{6.4}$$

Let (6.1) be in terms of M and p. After subtracting (6.1) from (6.4), we get:

$$L^{n,K}((2\overline{U}_j^k - U_j^k) - z_j^k) = p\overline{h}^2 + 2R_{2n}^k(\overline{x}_j), 1 \le j \le 2n-1 \tag{6.5}$$

or

$$L^{n,K}((2\overline{U}_j^k - U_j^k) - z_j^k) = 0(h^2), 1 \le j \le 2n-1.$$

Let

$$U_j^{ext,k} := 2\overline{U}_j^k - U_j^k.$$

$U_j^{ext,k}$ is another numerical approximation of z_j^k.

Using Lemma 4.2 we get the following result:

Theorem 6.1 *Let $U_j^{ext,k}$ be the numerical solution approximation, obtained* via *the Richardson extrapolation based on FOFDM* (4.5) *along with the boundary conditions* (4.2) *and z_j^k the solution to* (3.2) and (3.3) *at time level t_k. Then, there exists a constant M independent of ε, τ, h and k such that*

$$\max_{0 \le j \le n} |U_j^{ext,k} - z_j^k| \le M h^2. \tag{6.6}$$

Applying triangular inequality leads to

$$|U_j^{ext,k} - u_j^k| \le |U_j^{ext,k} - z_j^k| + |z_j^k - u_j^k|. \tag{6.7}$$

From Lemma 3.1, Theorem 6.1 and the global error, we get the following result.

Theorem 6.2 *Let $U_j^{ext,k}$ be the numerical solution of* (4.5) *along with the boundary conditions* (4.2) *and z_j^k the solution to* (1.1) and (1.2) *at the grid point* (x_j,t_k). *Then, there exists a constant M independent of ε, τ, h and k such that*

$$\max_{0\le j\le n} |U_j^{ext,k} - u_j^k| \le M(h^2 + \tau). \tag{6.8}$$

In the next section we implement the proposed scheme on two examples and present numerical results which confirm the accuracy and robustness of the solution.

7 | NUMERICAL EXAMPLES

In this section we present the numerical results of some problems of type (1.1) and (1.2).

Example 7.1 Consider the following singularly perturbed turning point problem

$$\left.\begin{aligned} \varepsilon u_{xx} + a(x,t)u_x - b(x,t)u - du_t = f(x,t), -1 \le x \le 1;\ \ \varepsilon, t \in [0,1],\\ u(-1,0) = u(1,0) = 0. \end{aligned}\right\} \tag{7.1}$$

where $d = 1$, $a(x,t) = 2x[1 + \sqrt{\varepsilon}t^2)]$ and $b(x,t) = 2(2 + xt)$.

This problem has an interior layer of width $\mathcal{O}(\varepsilon)$. The exact solution is

$$u(x,t) = \varepsilon(1 - x^2)\exp\left(-\frac{t}{\varepsilon}\right) erf\left(\frac{x}{\sqrt{\varepsilon}}\right).$$

To get the expression of $f(x,t)$ we substitute $a(x,t)$;$b(x,t)$ and $u(x,t)$ into Equation (7.1).

Example 7.2 Consider the following singularly perturbed turning point problem

$$\left.\begin{aligned} \varepsilon u_{xx} + a(x,t)u_x - b(x,t)u - du_t = f(x,t),\ \ 0 \le x \le 1,\ \ \varepsilon,\quad t \in [0,1],\\ u(0,0) = \varepsilon\tanh\left(\frac{1}{2\varepsilon}\right) - c;\quad u(1,0) = \varepsilon\tanh\left(-\frac{1}{2\varepsilon}\right) - c,\ \ c = \varepsilon^{\frac{2}{3}}, \end{aligned}\right\} \tag{7.2}$$

where $d = (1 + x^2)\exp(-t)$, $a(x,t) = 2(2x - 1)[1 + t^2)]$ and $b(x,t) = 2(1 + xt)$.

This problem has an interior layer of width $\mathcal{O}(\varepsilon)$. The exact solution is

$$u(x,t) = \varepsilon\exp\left[-\frac{t}{\varepsilon}\right]\tanh\left(\frac{0.5 - x}{\varepsilon}\right) - c\exp(-xt),$$

and $f(x,t)$ is obtained after substituting $u(x,t)$ into Equation (7.2).

The maximum errors at all mesh points and the numerical rates of convergence before extrapolation are evaluated using the formulas

$$E^{\varepsilon,n,K} := \max_{0\le j\le n;0\le k\le K}\left|U_{j,k}^{\varepsilon,n,K} - u_{j,k}^{\varepsilon,n,K}\right|.$$

In case the exact solution is unknown, we use a variant of the double mesh principle

$$E^{\varepsilon,n,K} := \max_{0\le j\le n;0\le k\le K}\left|U_{j,k}^{\varepsilon,n,K} - U_{j,k}^{\varepsilon,2n,2K}\right|.$$

where $u_{j,k}^{\varepsilon,n,K}$ and $U_{j,k}^{\varepsilon,n,K}$ in the above represent respectively the exact and the approximate solutions obtained using a constant time step τ and space step h. Similarly, $U_{j,k}^{\varepsilon,2n,2K}$ is found using the constant

TABLE 1 Maximum errors of Example 7.1 (before extrapolation)

ε	$N = 16$ $K = 10$	$N = 32$ $K = 20$	$N = 64$ $K = 40$	$N = 128$ $K = 80$	$N = 256$ $K = 160$
10^{-3}	6.34E−02	4.06E−02	2.25E−02	1.18E−02	6.05E−03
10^{-4}	6.34E−02	4.07E−02	2.26E−02	1.19E−02	6.09E−03
10^{-5}	6.34E−02	4.07E−02	2.26E−02	1.19E−02	6.09E−03
10^{-6}	6.34E−02	4.07E−02	2.26E−02	1.19E−02	6.10E−03
⋮	⋮	⋮	⋮	⋮	⋮
10^{-14}	6.34E−02	4.07E−02	2.26E−02	1.19E−02	6.10E−03

TABLE 2 Maximum errors of Example 7.2 (before extrapolation)

ε	$N = 16$ $K = 10$	$N = 32$ $K = 20$	$N = 64$ $K = 40$	$N = 128$ $K = 80$	$N = 256$ $K = 160$
10^{-3}	8.70E−02	4.71E−02	2.44E−02	1.24E−02	7.32E−03
10^{-4}	8.66E−02	4.69E−02	2.43E−02	1.23E−02	6.22E−03
10^{-5}	8.65E−02	4.68E−02	2.43E−02	1.23E−02	6.21E−03
10^{-6}	8.64E−02	4.68E−02	2.43E−02	1.23E−02	6.21E−03
⋮	⋮	⋮	⋮	⋮	⋮
10^{-14}	8.64E−02	4.68E−02	2.43E−02	1.23E−02	6.21E−03

TABLE 3 Rates of convergence of Example 7.1 (before extrapolation)

ε	r_1	r_2	r_3	r_4
10^{-3}	0.64	0.86	0.93	0.96
10^{-4}	0.64	0.85	0.93	0.96
10^{-5}	0.64	0.85	0.93	0.96
10^{-6}	0.64	0.85	0.93	0.96
⋮	⋮	⋮	⋮	⋮
10^{-14}	0.64	0.85	0.93	0.96

time step $\frac{\tau}{2}$ and space step $\frac{h}{2}$. Nevertheless, the computation of numerical rates of convergence is given by:

$$r_l = r_k \equiv r_{\varepsilon,k} := \log_2(E^{\varepsilon,n,K}/E^{\varepsilon,2n_l,2K_l}), \quad l = 1, 2, \ldots$$

Also, we compute $E_{n,K} = \max_{0<\varepsilon\leq 1} E_{\varepsilon,n,K}$.

And the numerical rate of uniform convergence is:

$$R_{n,k} := \log_2(E_{n,K}/E_{2n,2K}).$$

For a fixed mesh, we see that the maximum nodal errors remain constant for small values of ε (see Tables 1 and 2). Moreover, results in Tables 3 and 4 show that the proposed method is essentially first order convergent.

After extrapolation the maximum errors at all mesh points and the numerical rates of convergence are evaluated using the formulas:

$$E^{ext}_{\varepsilon,n,K} := \max_{0\leq j\leq 2n; 0\leq k\leq 2K} |U^{ext}_j - u^{\varepsilon,n,K}_{j,k}| \quad \text{and} \quad R_k \equiv R_{\varepsilon,k} := \log_2(E^{ext}_{n_k}/E^{ext}_{2n_k}), k = 1, 2, \ldots$$

respectively, where $E^{ext}_{n_k}$ stands for $E^{\varepsilon,2n,2K}$. Tables 5–8 confirm the theoretical predictions that Richardson extrapolation improves the accuracy of the numerical method employed and increases the rate of convergence.

TABLE 4 Rates of convergence of Example 7.2 (before extrapolation)

ε	r_1	r_2	r_3	r_4
10^{-3}	0.88	0.95	0.96	0.76
10^{-4}	0.88	0.95	0.98	0.99
10^{-5}	0.89	0.95	0.98	0.99
10^{-6}	0.89	0.95	0.98	0.99
⋮	⋮	⋮	⋮	⋮
10^{-14}	0.89	0.95	0.98	0.99

TABLE 5 Maximum errors of Example 7.1 (after extrapolation)

ε	$N = 16$ $K = 10$	$N = 32$ $K = 40$	$N = 64$ $K = 160$	$N = 128$ $K = 640$	$N = 256$ $K = 2{,}560$
10^{-3}	8.87E−02	2.65E−02	5.25E−03	9.45E−04	6.06E−04
10^{-4}	8.98E−02	2.97E−02	7.88E−03	1.68E−03	3.05E−04
10^{-5}	8.99E−02	2.97E−02	8.08E−03	2.06E−03	4.89E−04
10^{-6}	8.99E−02	2.98E−02	8.09E−03	2.07E−03	5.19E−04
⋮	⋮	⋮	⋮	⋮	⋮
10^{-14}	8.99E−02	2.98E−02	8.09E−03	2.07E−03	5.20E−04

TABLE 6 Rates of convergence of Example 7.1 (after extrapolation)

ε	r_1	r_2	r_3	r_4
10^{-3}	1.74	2.34	2.47	0.64
10^{-4}	1.60	1.91	2.23	2.46
10^{-5}	1.60	1.88	1.97	2.07
10^{-6}	1.60	1.88	1.97	1.99
⋮	⋮	⋮	⋮	⋮
10^{-10}	1.60	1.88	1.97	1.99

TABLE 7 Maximum errors of Example 7.2 (after extrapolation)

ε	$N = 16$ $K = 10$	$N = 32$ $K = 40$	$N = 64$ $K = 160$	$N = 128$ $K = 640$	$N = 256$ $K = 2{,}560$
10^{-3}	1.07E−01	3.10E−02	8.00E−03	7.32E−03	7.32E−03
10^{-4}	1.07E−01	3.13E−02	8.21E−03	2.07E−03	1.46E−03
10^{-5}	1.07E−01	3.13E−02	8.21E−03	2.08E−03	5.21E−04
10^{-6}	1.07E−01	3.13E−02	8.21E−03	2.08E−03	5.21E−04
⋮	⋮	⋮	⋮	⋮	⋮
10^{-14}	1.07E−01	3.13E−02	8.21E−03	2.08E−03	5.21E−04

TABLE 8 Rates of convergence of Example 7.2 (after extrapolation)

ε	r_1	r_2	r_3	r_4
10^{-3}	1.77	1.93	1.99	0.50
10^{-4}	1.77	1.93	1.98	2.00
10^{-5}	1.77	1.93	1.98	2.00
10^{-6}	1.77	1.93	1.98	2.00
⋮	⋮	⋮	⋮	⋮
10^{-14}	1.78	1.93	1.98	2.00

8 | CONCLUDING REMARKS AND SCOPE OF FUTURE RESEARCH

Singularly perturbed turning point problems are difficult to solve using standard/classical methods due to the presence of boundary or interior layers in their solutions. Usually, when seeking for numerical solutions of layer problems, layer adapted meshes are used. These meshes are fine in the layer region and coarse away from the layer region. Due to the nature of these meshes, and especially when time is involved, the computation with regards to the convergence analysis becomes more complex.

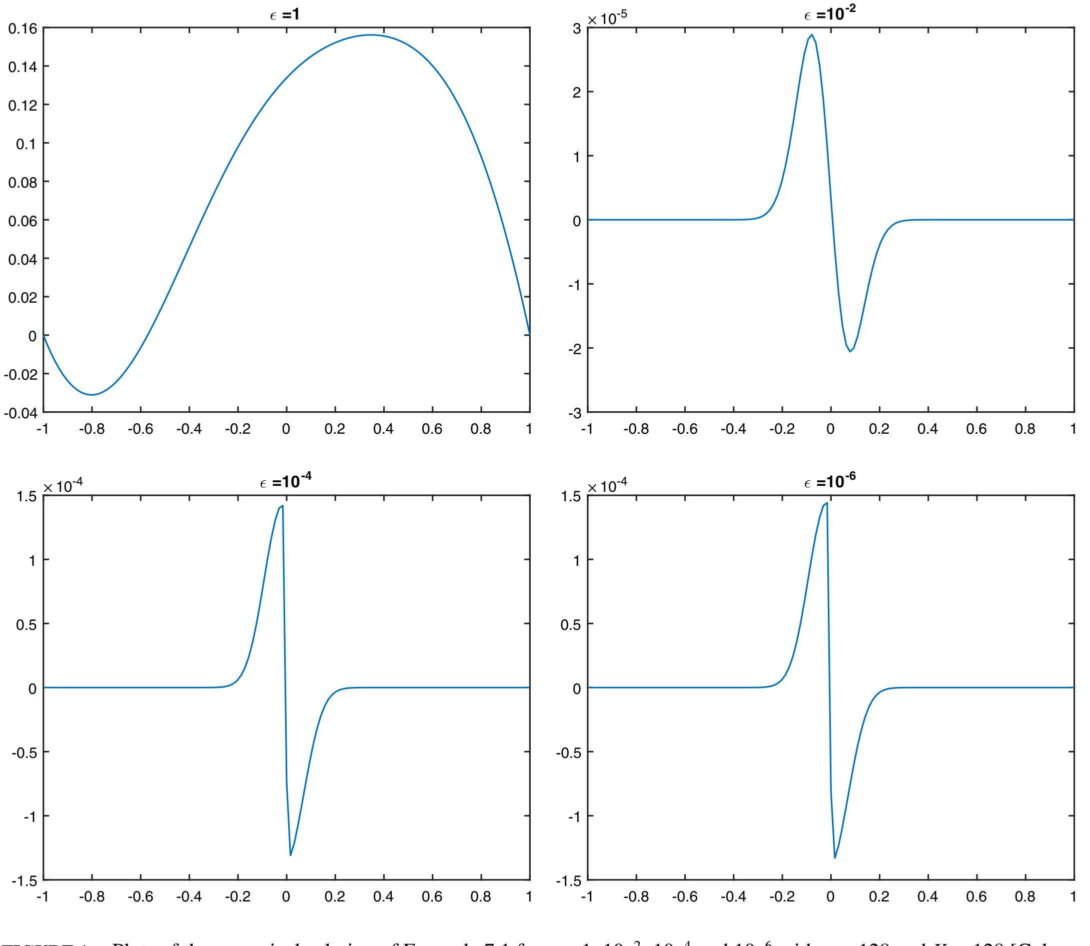

FIGURE 1 Plots of the numerical solution of Example 7.1 for $\varepsilon = 1$, 10^{-2}, 10^{-4} and 10^{-6} with $n = 128$ and $K = 128$ [Color figure can be viewed at wileyonlinelibrary.com]

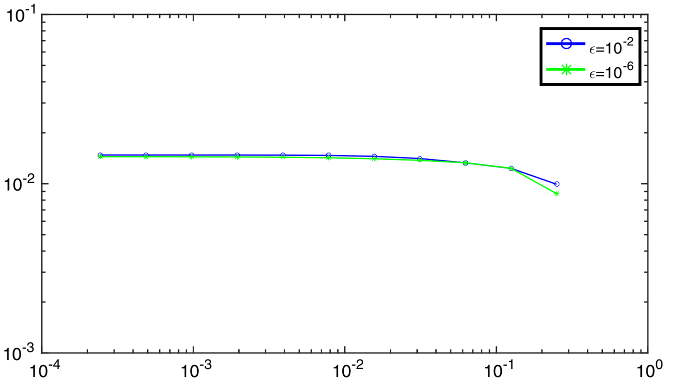

FIGURE 2 Log–log plot for Example 7.2: The logarithm of pointwise maximum errors is plotted against the logarithm of step size h at time $t = 1$ with values of n from 4 to 4,096 and for $\varepsilon = 10^{-2}$ and 10^{-6} [Color figure can be viewed at wileyonlinelibrary.com]

The main aim of this paper was to design and analyze a FOFDM to solve a class of time dependent singularly perturbed turning point problems whose solution exhibits an interior layer. We first established bounds on the solution and its derivatives. Then, we discretized the time variable before proceeding to space discretization. Bounds were used to prove uniform convergence of the proposed numerical method. The first order uniform convergence shown theoretically, with respect to space and time variables was confirmed numerically through two test examples.

We provided plots of the numerical solution for Example 7.1 for various values of the perturbation parameter ε to see the layer behavior (see Figure 1). In addition, we presented a log–log plot for Example 7.2 (see Figure 2).

We also applied Richardson extrapolation to improve the accuracy and the convergence of the numerical scheme in the space variable. Indeed, convergence order improved from one to two.

The problem investigated in this paper depends on the perturbation parameter ε which multiplies the highest order derivative that appears in the problem. One would like to understand how replacing ε by some function of ε and x affects the design of numerical methods. We are currently working in that direction.

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